Unawareness and Framing*

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April 2006

Abstract

We introduce a model of unawareness founded on preferences over descriptions of acts. The model highlights the effects of how events in the state space are framed by distinguishing different descriptions of the same act. A more detailed description immediately confers a higher level of awareness to the decision maker. The primitive is a family of preferences, indexed by partitions of the state space. Each partition corresponds to a description of the state space. We axiomatically characterize the following partition-dependent expected utility representation. The decision maker has a nonadditive set function over events. She then computes expected utility with respect to her partition-dependent belief, which weights explicitly listed events. Unawareness can be expressed through betting preference and subjective likelihood rather than through knowledge. Absolute and relative notions of unawareness and response to unawareness are presented.

1 Introduction

Consider a newly hired worker comparing available health insurance plans during open enrollment. While she understands some broad possible contingencies, like requiring a surgery or becoming pregnant, she is unaware of more specific contingencies, like requiring a laminotomy.[1] Despite her partial awareness of the environment, the worker must decide on some health plan before the end of the enrollment period. How much is she willing to pay for the different insurance options? Can an outside observer, who knows what kind of coverage she will purchase, distinguish the contingencies of which the employee is aware from those of which she has never heard? Can the observer distinguish those of which the employee has never heard from those that she believes are impossible? Can the observer place predictive restrictions on the employee's behavior as she becomes aware of more contingencies?

*We thank the National Science Foundation for financial support under Grant SES-0550224.

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[1] Incidentally, a laminotomy is a surgery which removes a thin bony layer covering the spinal canal.

This paper introduces a novel methodology for answering such questions. It uses the framing of acts to elicit the decision maker's response to different levels of awareness. For example, consider the following contract, which associates deductibles on the left with contingencies on the right:

$$\begin{pmatrix} \$500 & \text{surgery} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Compare this to the following contract, which includes some redundancies:

$$\begin{pmatrix} \$500 & \text{laminotomy} \\ \$500 & \text{other surgeries} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Both contracts represent the same effective levels of coverage, but the worker might evaluate these lists differently, because the second formulation makes her aware of laminotomies. If she is willing to pay more for the second contract, this disparity might suggest unawareness of laminotomies when she had been presented the first contract. Her unawareness while evaluating the first, more coarsely expressed contract reveals itself when she is willing to pay more for the second, more finely expressed contract. This difference perhaps reflects an updated and increased personal belief of the likelihood of surgery. The broader conceptual point is that the presentation and expression of an act immediately confers some information to the reader about the structure of the state space. Conversely, the decision maker must at least understand the coarsest partition required to express the structure of the contact. Moreover, increasingly refined expressions of the contract must confer correspondingly more awareness. We assume this prerequisite or conferred awareness on behalf of the decision maker, and use it to identify features of her behavior. To our knowledge, this paper is the first axiomatic attempt to connect the measurability and expressive requirement of an act to the decision maker's awareness. It attempts to provide a unified treatment of awareness and framing.

The following example perhaps illustrates the relationship between awareness and framing more sharply. The mathematician Jean d'Alembert argued that "the probability of observing at least one head in two tosses of a fair coin is 2/3 rather than 3/4. Heads, as he said, might appear on the first toss, or, failing that, it might appear on the second, or, finally, might not appear on either. D'Alembert considered the three possibilities equally likely (Savage 1954, p. 65)." D'Alembert's fundamental mistake was in his framing of the states; he failed to split the first event into its two atoms: heads then tails, and heads then heads. He viewed the world as three events: $\{HH, HT\}$, $\{TH\}$, and $\{TT\}$. Had he been aware of the two subevents comprising $\{HH, HT\}$ and framed the possible tosses appropriately, he may have avoided the error.

Such framing effects are precluded in the standard models of decision making under uncertainty introduced by Savage (1954) and by Anscombe and Aumann (1963). These models do not distinguish between different presentations of the same act, implicitly assuming that the framing of the state space is inconsequential. We introduce a richer set of primitives which treats the different frames for an act as distinct choice objects.

In particular, our model treats lists of contingencies and outcomes as the primitive objects of choice. The following list

$$\begin{pmatrix} x_1 & E_1 \\ x_2 & E_2 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix},$$

denotes an act which delivers the outcome x_i if the state of the world is in E_i. In papers, acts are often denoted by such lists for ease of exposition. We take these expressions literally. For example, if $E_1' \cup E_1'' = E_1$, then the following list

$$\begin{pmatrix} x_1 & E_1' \\ x_1 & E_1'' \\ x_2 & E_2 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix},$$

denotes the same act, but is modeled as a distinct object. The decision maker might have different attitudes about the two presentations, because the second has made her aware of the more specific contingencies E_1' and E_1''. This discrimination between lists is the primary methodological innovation of the paper.

Such discrimination also provides a natural framework for studying how the decision maker updates her preferences as she becomes aware of possibilities she did not previously understand. For example, what would d'Alembert have done had he realized that HH is distinct from HT? Can we make any predictive restrictions on his behavior after this realization from his preferences before the realization? One can view the different lists as reflective of more or less awareness, and the choices between lists are reflective of the corresponding preferences. The response to new awareness is central to our approach. In fact, since our proposed representation involves expected utility to aggregate uncertainty at a fixed enumeration of states, comparisons of beliefs across descriptions are the only way to identify unawareness from preference.

Aside from theoretical concerns, many real contracts are presented as such lists. Insurance plans are often described by a table of contingencies and coverage amounts. Table 1 is a partial verbatim copy of a Blue Cross medical plan available to University of California employees expressed as procedures and deductibles. The other available plans are similarly described.

We study a decision maker who acts as if she places a weight $\nu(E)$ on each event E. When presented a description $E_1, \ldots, E_n$ of the possibilities, she judges the probability of E_i to be

Office visit	\$20
Hospital visit	no charge
Preventive physical exam	\$20
Maternity outpatient care	\$20
Maternity inpatient care	\$250

Table 1: Blue Cross health insurance plan

$\nu(E_i)/\sum_j \nu(E_j)$. Since the weighting function ν is not necessarily additive, her probability of E_1 can depend on whether it is expressed as E_1 or expressed as $E_1' \cup E_1''$. Her utility for a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

is obtained by aggregating her cardinal utilities $u(x_i)$ over the consequences x_i by the weights $\nu(E_i)/\sum_j \nu(E_j)$ on their corresponding events E_i. Then the nonadditivity of ν can be used to measure and compare the awareness of E_1' when she is only told of E_1. So, although the primitives are richer, our proposed utility representation maintains the essential notions of expected utility and subjective probability from the standard model.

The articulation of awareness in the decision theoretic terms of betting and likelihood yields several benefits. First, it provides a language to discuss partial awareness of unawareness. While having nothing directly to say about interactive epistemology, we hope our work complements a sizable literature studying semantic models which pose awareness and unawareness as epistemic operators on events.[2] Often, it either begins with or arrives at the position that the decision maker should have no awareness of her unawareness. For example, Dekel, Lipman, and Rustichini (1998, p. 161) argue that "an agent who is unaware of a possibility should have *no* positive knowledge of it at all." This claim was cast in the context of possibility correspondences in epistemic models, where the dichotomous nature of the awareness operator hinders the expression of partial awareness in the required terms. Either a decision maker is aware of an event or she is not. Then she is either aware of her unawareness or she is not. In this sense, the severity of the claim is at least partially an artifact of modeling choices.

While concurring that one should never be completely aware of her unawareness, our intuition departs before the severity conclusion that she should have no awareness at all. For example, a consumer may be partially aware that she has a less than complete medical understanding of her health or an investor may be partially aware that he cannot consider the universe of all mutual funds. What they cannot do is express the diseases or the mutual funds they don't know exactly. By expressing awareness in terms of betting and likelihood, rather than knowledge, we hope to introduce some notion of partial awareness, hence also a notion of partial awareness of unawareness.

[2]Feinberg (2004) and Heifetz, Meier, and Schipper (2005) provide syntactic analyses of interactive unawareness.

In our model, partial awareness is not a part of the description of the state space, but identified through comparisons of beliefs between descriptions. For example, we suggested at the beginning of the introduction to compare the consumer's willingness to pay for insurance contracts presented with varying levels of descriptive detail to determine her awareness.

Some recent work expands the basic semantic structure to include different projections or partitions of state spaces to reflect different levels of awareness (Heifetz, Meier, and Schipper forthcoming, Li 2006). But, at a fixed state space, the austere epistemic model still provides no channel for the decision maker to express a partial conception of the more refined semantics to which he does not have immediate access. These different state spaces might also be viewed as different frames of a single space. Under this interpretation, one contribution of this paper is to introduce an axiomatic notion of choice into these enriched structures.

However, this connection is strained by the following major difference between these papers and ours. Heifetz, Meier, and Schipper (forthcoming) and Li (2006) treat each state space as the fixed awareness of the decision maker; she does not have any more awareness. On the other hand, we treat each frame as the *minimal* awareness that the analyst can attribute to the decision maker, based on the expression of the acts. We then let the choice behavior provide some insight on her awareness beyond this minimal level. Because the representation involves likelihoods, it provides another channel to express partial awareness. Specifically, as alluded to in the beginning of this introduction, the disparity in subjective beliefs between different frames will gauge the decision maker's partial awareness. For example, if there is no change in these beliefs, then the decision maker acts as if she is completely aware of the more refined state space.

We also hope this model complements the existing decision theoretic work on unforeseen contingencies. Kreps (1979) introduced an axiomatic model of preference for flexibility by considering menus of objects as the primitives of choice. Demand for flexibility is interpreted as a response to unforeseen contingencies, which are captured in the proposed representation as subjective taste uncertainties. Dekel, Lipman, and Rustichini (2001), henceforth DLR, extend this approach to menus of lotteries, where the linear structure essentially identifies the subjective state space.[3] Then the analyst can remarkably determine the space of uncertainty as a theoretical artifact of preference, rather than assume a state space a priori.

The DLR methodology provides a powerfully unified treatment of states, beliefs, and utilities. On the other hand, because it depends on the decision maker's preferences to elicit the states, recovering unawareness is difficult, encountering the basic conundrum that the decision maker cannot reveal something of which she is totally unaware. In fact, in DLR's main representations, the decision maker acts as if she has complete awareness of some state space.

Epstein and Marinacci (2005) propose a generalization of DLR to address this conundrum. They suggest a form of maxmin expected utility, similar to Gilboa and Schmeidler's (1989) theory of ambiguity aversion, over the likelihood of the subjective states as a response to the decision maker's partial awareness that her understanding of the world is coarse. This resonates with

[3] Dekel, Lipman, Rustichini, and Sarver (2005) report a technical corrigendum to the original paper.

work by Ghirardato (2001), Mukerji (1996), and Nehring (1999) who capture the decision maker's partial awareness of unforeseen contingencies through Choquet integration of belief functions or capacities, which is also used to model ambiguity by Schmeidler (1989). One point of this paper is to demonstrate another method of detecting unawareness without invoking ambiguity aversion. In fact, our representation satisfies the standard expected utility axioms for each fixed description of the state space.

One way to distinguish DLR's approach and ours is that DLR relax Savage's assumption that the analyst has a complete understanding of the state space and forward a model where the states of the world are revealed through preference. Then unawareness can only be elicited through violations of expected utility. We equip the model, hence the analyst, with a comprehensive understanding of the state space. This means that the state space is not a part of the representation in the model. This assumption is a strong one, but we hope that it is justified by its conceptual dividends. It also seems more palatable when imagining applications. For example, in interactions, one party often has a superior understanding of the possible contingencies. Then the complete state space of our model only has to correspond to the understanding of the most aware agent in situations of asymmetric awareness, rather than to some objective or physical specificity. It also seems difficult to verify and enforce contracts contingent on DLR's subjective states, which are conceptual artifacts of the theory; such contracts are analogous to insurance against discount factors or coefficients of risk aversion. Insofar as unawareness and unforeseen contingencies bear on contracting, assuming some sort of objective state space seems less heroic. In fact, one motivation for developing this model is to accommodate unawareness in a Savage setting without invoking menus or multi-valued consequences, so the primitives bear as close a resemblance as possible to the way that actual contracts, like insurance policies or warranties, are presented.

Finally, we believe that the framing interpretation of the model is of its own interest and importance, independently of its application to awareness. Economists now appreciate the framing effects of consequences, especially as formalized by prospect theory (Kahneman and Tversky 1979), at both a theoretical and an applied level. On the other hand, the framing effects of contingencies seem relatively obscure. This is despite a sizable psychological literature documenting violations of extensionality, the psychological term for the invariance of the judged probability of an event to its particular expression. For example, Fischoff, Slovic, and Lichtenstein (1978) find that car mechanics' believe that a particular automobile component is much more likely to fail if the component's subparts are explicitly listed. It is difficult to attribute this distortion to unawareness on the part of the mechanics. Tversky and Koehler (1994) propose a theory of judgment, which they coin support theory, with many similarities to the theory of decision forwarded here. One contribution of this paper is to provide an axiomatic foundation for a generalized version of support theory. While the connections are discussed in the sequel, we should note now that many of the behavioral intuitions of the model should be credited to this psychological literature in general and to Tversky and Koehler (1994) in particular. We hope this paper directs more economic attention to how the framing of contingencies influences the judgment of likelihood, which we think is a significant psychological finding with potentially important economic consequences.

In the next section, we introduce the primitives of our theory. We then propose a utility representation for the model and provide an axiomatic characterization. Finally, we suggest methods for detecting correction for unawareness and comparing this correction across individuals.

2 A model of decision making with unawareness and framing

We introduce our formal model. Let S denote an arbitrary state space, which captures all the relevant uncertainty in the world. Let X denote the finite set of consequences or prizes. We invoke the Anscombe–Aumann structure, so let ΔX denote the set of all lotteries on consequences. An Anscombe–Aumann act $f : S \to \Delta X$ assigns an objective lottery $f(s) \in \Delta X$ to each state $s \in S$. The act f is simple if it takes finitely many values, $|\{f(s) : s \in S\}| < \infty$. Let $\mathcal{F}$ denote the space of all simple Anscombe–Aumann acts. Slightly abusing notation, and let $p \in \Delta X$ also denote the corresponding constant act which assigns p to every state. Let Π denote the collection of all finite partitions of S. For any particular partition $\pi \in \Pi$, let $\sigma(\pi)$ denote the algebra induced by π. Since π is finite, its induced algebra $\sigma(\pi)$ is the family of unions of cells in π and the empty set. Let $\mathcal{F}_\pi = \{f \in \mathcal{F} : f \text{ is } \sigma(\pi)\text{-measurable}\}$ denote the acts which respect the partition π, or such that $f^{-1}(p) \in \sigma(\pi)$ for all $p \in \Delta X$. In words, the act f if $\sigma(\pi)$-measurable if it assigns a constant lottery to all states in a particular cell of the partition: if $s, s' \in E \in \pi$, then $f(s) = f(s')$. We consider a *family* of preferences $\{\succsim_\pi\}_{\pi \in \Pi}$ indexed by partitions π, where each preference relation $\succsim_\pi$ is defined over π-measurable acts. If either f or g is not π-measurable, then the statement $f \succsim_\pi g$ is nonsensical. The strict and symmetric components $\succ_\pi$ and $\sim_\pi$ carry their standard meanings. For any act $f \in \mathcal{F}$, let $\pi(f)$ denote the minimal expression of f, which is the coarsest partition of S such that $f \in \mathcal{F}_\pi$. Similarly for any pair of acts $f, g \in \mathcal{F}$, let $\pi(f, g)$ be the coarsest partition such that $f, g \in \mathcal{F}_\pi$. Given a partition $\pi = \{E_1, \ldots, E_n\}$ and acts $f_1, \ldots, f_n \in \mathcal{F}$ define a new act by:

$$\begin{pmatrix} f_1 & E_1 \\ \vdots & \vdots \\ f_n & E_n \end{pmatrix}(s) = \begin{cases} f_1(s) & \text{if } s \in E_1 \\ \vdots & \vdots \\ f_n(s) & \text{if } s \in E_n \end{cases}.$$

The following definition adapts the standard concept of null events for our setting with a family of preferences.

Definition 1. Given $\pi \in \Pi$, an event $E \in \sigma(\pi)$ is **π-null** if

$$\begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \sim_\pi \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix},$$

for all $f \in \mathcal{F}_\pi$ and $p, q \in \Delta X$. $E \in \sigma(\pi)$ is **π-nonnull** if it is not π-null. The event E is **null** if $E = \emptyset$ or if E is π-null for any π such that $E \in \pi$. E is **nonnull** if it is not null.

This family of preferences might not immediately appear to be related to our original motivation of studying lists. In fact, this family provides a parsimonious primitive which is isomorphic to a model which begins with preferences over lists. Suppose we started with a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

which is a presentation of the act f. This could be more compactly represented by a pair (f, π), where the partition $\pi = \{E_1, \ldots E_n\}$ denotes the list of contingencies on the right hand side of the list. This description π must be at least rich enough to describe the act f, so we can assume $f \in \mathcal{F}_\pi$. Now suppose the decision maker is deciding between two lists, which are represented as (f, π_1) and (g, π_2). Since she has read both descriptions, she is now aware of both π_1 and π_2; alternatively, she must be aware of both π_1 and π_2 to compare the lists. She is aware of any $E \in \pi_1$ from reading the first list and of $F \in \pi_2$ from reading the second. Then her minimal level of awareness is the coarsest common refinement of π_1 and π_2, their join $\pi = \pi_1 \vee \pi_2$. Then (f, π_1) is preferred to (g, π_2) if and only if (f, π) is preferred to (g, π). So, we can restrict attention to the preference restricted to pairs (f, π) and (g, π) where $f, g \in \mathcal{F}_\pi$. Moving the partition from being carried by the acts to being carried as an index of the preference relation results exactly in the model being studied here. We stress that the model is really that of a decision maker deciding between lists. The lists are expressed through indexed preference relations for the resulting economy of notation, which will simplify understanding the technical mechanics of the model.

For example, suppose the decision maker is deciding between the Blue Cross health plan described on Table 1 and the healthy plan available from Kaiser Permanente and depicted in Table 2. The partitioning of the Kaiser Permanente plan differs from the partitioning of the Blue Cross

Primary and specialty care visits	\$50
Well-child visits to age two	\$15
Family planning visits	\$50
Scheduled prenatal care and first postmartum visit	\$50
Maternity inpatient care	\$250

Table 2: Kaiser Permanente health insurance plan

plan; some contingencies are explicitly listed on one list but not on the other. A newly hired and naive assistant professor, in the process of comparing health insurance options, becomes aware of the possibility of maternity outpatient care, from the Blue Cross plan, and of family planning visits, from the Kaiser Permanente plan. This level of awareness is a consequence of simply reading the lists and merely reflects her exposure to both policies. So, even though neither contract is as fine as $\pi(f, g)$, since the decision maker is reading both, we feel comfortable attributing $\pi(f, g)$ as her minimal amount of awareness in considering the two acts.

By assuming $f, g \in \mathcal{F}_\pi$, we assure that π is at least as fine as $\pi(f, g)$. This interpretation highlights an important interpretive difference between standard theories of Bayesian updating and our theory of awareness. In the former, functions must be restricted to respect the information, or lack thereof, embodied in an algebra on the state space. In our model, it is the algebra that must be expanded to reflect the awareness implicit in the description of an act.

Partitions or subalgebras are often used to model the arrival of information about the actual state of the world, where each cell of a partition represents an updated restriction on the truth. Our interpretation is quite different. We take each partition π as a frame of awareness or a description of the entire state space. Each cell represents an event that a decision maker understands and of which she is aware. For example, she may be aware that her car may break down, yet be unaware that one of the ways it might break down is a sudden disintegration of the tires. In our model, she does *not* learn at some ex interim stage which particular cell actually obtains, i.e. whether her car will actually break down in the future because of tire disintegration or for some other reason.

One feature of our model is that the decision maker's minimal awareness is formalized with respect to partitions. We feel that focusing on the awareness of an entire partition of the state space is superior to discussing the awareness of particular states or events. A similar view is expressed in semantic models with lattices or partitions of state spaces by Heifetz, Meier, and Schipper (forthcoming) and Li (2006) and in the interpretation of DLR's subjective state space by Epstein and Marinacci (2005), who suggest that the term "coarse contingencies" is more evocative than "unforeseen contingencies." For example, an investor may have been unaware of the possibility of domestic terrorism before September 11, 2001. Afterwards, she updates her awareness. The investor does not become aware of a new state of the world, because a terrorist action does not constitute a full resolution of relevant uncertainty. She is still concerned with the prime interest rate, the price of oil, and all the other variables that priced her investments before she was aware of terrorism. Rather than becoming aware of a single state, she becomes aware that each cell she had previously considered a full resolution of the relevant uncertainty had actually been incomplete.

Graphically, suppose the investor's view of the world before September 11 was:

s_1	s_2	s_3

If we model her awareness of terrorism as a new state t, then her new view would look like:

s_1	s_2	s_3	t

If we model her awareness as a new *partition* which filtrates terrorism (t) and no terrorism (n), her new view has six cells and would look like:

s_1, t	s_2, t	s_3, t
s_1, n	s_2, n	s_3, n

We believe the latter view is much more aligned with the general process of updating awareness.

A model which considers only awareness of states also has difficulties accommodating a decision maker who is aware of an event, but unaware of its components. For example, it seems reasonable that a consumer might be aware of the fact that her car might break down, and even have a well-formed probability of this event, yet have only a vague idea of the different components in her car that might fail. In the standard epistemic expressions of unawareness, awareness of an event implies awareness of its subevents. By using partitions, our models separates awareness from set inclusion, since a partition can obviously be fine enough to include an event but be too coarse to include its subevents.

3 Partition-dependent expected utility

We propose the following utility representation for every $\succsim_\pi$. The decision maker has a nonnegative set function $\nu : 2^S \to \mathbb{R}_+$ over all events. When she is presented with an enumerated description $\pi = \{E_1, E_2, \ldots, E_n\}$ of the state space, she places a weight $\nu(E_k)$ on each event. Normalizing these weights by their sum, $\mu_\pi(E_k) = \nu(E)/\sum_i \nu(E_i)$ defines a probability measure μ_π over $\sigma(\pi)$, the algebra induced by π. Then, her utility for the act f expressed as:

$$f = \begin{pmatrix} p_1 & E_1 \\ p_2 & E_2 \\ \vdots & \vdots \\ p_n & E_n \end{pmatrix},$$

is $\sum_{i=1}^n u(p_i)\mu_\pi(E_i)$, where $u : \Delta X \to \mathbb{R}$ is an affine von Neumann–Morgenstern utility function on objective lotteries over consequences.

The following restriction avoids division by zero when normalizing the set function.

Definition 2. A set function $\nu : 2^S \to \mathbb{R}_+$ is *nondegenerate* if (i) $\nu(\emptyset) = 0$ and (ii) $\sum_{E\in\pi} \nu(E) > 0$ for all $\pi \in \Pi$.

We can now formally define our suggested utility representation.

Definition 3. $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a **partition-dependent expected utility** representation if there exist a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ and a nondegenerate positive set function $\nu : 2^S \to \mathbb{R}_+$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \iff \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi,$$

where μ_π is the unique probability measure on $(S, \sigma(\pi))$ such that, for all $E \in \pi$:

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F\in\pi} \nu(F)}. \tag{1}$$

When such a pair (u, ν) exists, we call it a partition-dependent expected utility representation.

The set function ν in this definition is not necessarily additive, and its nonadditivity provides a channel for detecting the decision maker's awareness of events which are more specific than those included in the description of the act. While the representation involves nonadditive set functions, it is only superficially similar to Choquet expected utility (Schmeidler 1989). In fact, the decision maker acts as if she is probabilistically sophisticated at each partition π. The structure of ν is quite general: ν is not necessarily monotone nor convex-ranged. It can also be strictly bounded away from zero for nonempty events, in which case there are no null events, even if the state space is uncountably rich.

A partition-dependent expected utility representation provides the following guidelines for the decision maker's response to unawareness. Each event $E \subsetneq S$ carries a value $\nu(E)$, which corresponds to its relative weight in frames where the decision maker must be aware of E but not necessarily of its subevents. The nonadditivity of ν captures the effects of framing or unawareness: A and B can be disjoint yet $\nu(A)+\nu(B) \neq \nu(A \cup B)$. If π_E is a partition of E, then the difference $\sum_{F\in\pi_E} \nu(F) - \nu(E)$ captures the unawareness of π_E relative to E. Of course, if ν is additive, then the decision maker acts as if she is totally aware of the state space and her behavior corresponds to standard Bayesian updating. Moreover, she may have complete awareness over part of the state space, i.e. if ν is additive over all the subevents of E, without awareness over the entire state space, i.e. if ν is nonadditive over subevents of $E^{\complement}$. In the representation, unawareness departs from Bayesian decision theory only in the *response* to new awareness or information, but not in the choices within a fixed mode of awareness, since each $\succsim_\pi$ conforms to expected utility. From the analyst's perspective, the detection or elicitation of unawareness therefore hinges on this response, on the dynamics between partitions. If she accepts our axioms, the analyst can predict the decision maker's behavior in frame π from her behavior in other frames.

Notice that the inequality $\sum_{F\in\pi_E} \nu(F) < \nu(E)$ is not precluded. In words, the decision maker could put less weight on an event as she becomes aware of more specifics. This is because ν does not capture unawareness in isolation, but also reflects the decision maker's correction for her unawareness. She may be partially aware that her conception of the event E is incomplete, and try to incorporate what "she believes she does not know" into her odds. If she overcompensates for her unawareness, the resulting $\nu(E)$ might be larger than the sum of its components. For example, a car owner might understand that there are myriad ways for her car to break down, but can name only a few. One plausible response might be to purchase more insurance than would be optimal had she possessed a full mechanical understanding of her car.

One of the attractive features of the proposed representation is it compact form. Like standard Anscombe–Aumann expected utility, preference is summarized by two functions, one for utility and another for likelihood. Its small departure from standard theory, dropping additivity, allows for a much richer class of behaviors. In the special case that the set function ν is additive, the probabilities of events do not depend on their expressions. Then the decision maker is indistinguishable from someone who has full awareness of the state space.

Definition 4. $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a **partition-independent expected utility** representation if there exist a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ and a finitely additive probability measure $\mu : 2^S \to [0, 1]$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \Longleftrightarrow \int_S u \circ f \, d\mu \geq \int_S u \circ g \, d\mu.$$

Tversky and Koehler (1994) introduced a related nonextensional theory of judgement called support theory. Its primitives are different descriptions of events, called hypotheses. It analyzes binary comparisons of likelihood between two hypotheses, which they call evaluation frames, which consist of a focal hypothesis and an alternative hypothesis. The probability judgment of the focal hypothesis A relative to the alternative B in the evaluation frame (A, B) is proposed to be $P(A, B) = s(A)/[s(A) + s(B)]$, where $s(A)$ is the support assigned to hypothesis A, based on the strength of its evidence. They offer a characterization of such judgments based on functional equations, but this characterization is primarily technical and not founded on preference (Tversky and Koehler 1994, Theorem 1). Our theory translates support theory from judgment to decision making, extends its scope beyond binary evaluation frames, and provides an axiomatic foundation from preference. The motivation is also quite different, since Tversky and Koehler attribute violations of extensionality to heuristic devices like availability, where the decision maker judges probabilities by her ability to recall typical cases, rather than to unawareness. They also present extensive experimental evidence illustrating the sensitivity of judgments of probability to the description and framing of the possibilities.

The following special cases provide some intuition for partition-dependent expected utility.

Example 1 (Probability weighting)**.** Suppose $\nu(E) = \varphi(\mu(E))$, where $\mu : 2^S \to \mathbb{R}$ is a finitely additive probability measure and $\varphi : [0, 1] \to [0, 1]$ is a strictly increasing transformation with $\varphi(0) = 0$ and $\varphi(1) = 1$. In the theory of Choquet integration and nonadditive beliefs, such transformations are sometimes called probability distortion functions. In theories of non-expected utility over objective lotteries, like prospect theory (Kahneman and Tversky 1979) and anticipated utility theory (Quiggin 1982), such transformations are called probability weighting functions.[4] Their application in our model is closest to the subjectively weighted utility theory of Karmarkar (1978). As Quiggin (1982) points out, because it is independent of the consequences tied to the lottery, the weighting function φ in subjectively weighted utility must be linear, otherwise the preference violates stochastic dominance. Here, because the manner in which φ is applied depends on the framing of the act, we finesse this reduction and provide a nontrivial expression of subjectively weighted utility satisfying stochastic dominance.

This example illuminates a framing dependence of objective theories which depend on weighting functions . Instead of working with the space of probability distributions or lotteries, suppose the objects of choice were lists of outcomes and odds, analogous to the lists of outcomes and events considered here. If a list included redundancies, for example if the probability p of x was broken into $p_1 + p_2 = p$, then a nonlinear weighting function would aggregate the redundant expression

[4] In some applications, φ is assumed to be continuous.

differently from the minimal expression.

Example 2 (Principle of insufficient reason)**.** Suppose ν is a constant function, for example $\nu(E) = 1$ for each every nonempty E. Then the decision maker puts equal probability on all the events of which she is cognizant. Such a criterion for cases of extreme ignorance or unawareness was advocated by Laplace as the principle of insufficient reason. This principle is sensitive to the framing of the states. Consider the error of d'Alembert mentioned in the introduction, who attributed a probability of 2/3 to seeing at least one head among two tosses of a fair coin. The fundamental error was his framing of the state space as H-$, TH, TT$, or partitioned as $\{\{HT, HH\}; \{TH\}; \{TT\}\}$. If ν is constant, d'Alembert would have realized his error had he been presented a bet which pays only on a head followed by a tail, HT. On the other hand, he would have made a similar error had he reasoned that there can be either 0, 1, or 2 heads, partitioning the states into $\{\{TT\}; \{HT, TH\}; \{HH\}\}$. This criticism is difficult to even formalize in a standard decision model; ours is specifically designed to capture such framing effects.

A more tempered resolution of unawareness is a convex combination of the ignorance prior and a probability measure μ: $\nu(E) = \alpha + (1-\alpha)\mu(E)$ for some $\alpha \in [0, 1]$. Fox and Rottenstreich (2003) report experimental evidence which suggests that judgment is partially biased towards the ignorance prior.

4 Axioms and characterizations

We now provide axiomatic characterizations of both partition-dependent and partition-independent expected utility. We also discuss the somewhat subtle uniqueness of ν, which requires an additional condition.

4.1 Representations

The first five axioms on preference essentially apply the standard Anscombe–Aumann axioms to each $\succsim_\pi$. We will refer to Axioms 1–5 collectively as the Anscombe–Aumann axioms.

Axiom 1 (Preference)**.** $\succsim_\pi$ is complete and transitive for all $\pi \in \Pi$.

Axiom 2 (Independence)**.** For all $f, g, h \in \mathcal{F}_\pi$ and $\alpha \in (0, 1)$: if $f \succ_\pi g$, then $\alpha f + (1-\alpha)h \succ_\pi \alpha g + (1-\alpha)h$.

Axiom 3 (Archimedean Continuity)**.** For all $f, g, h \in \mathcal{F}_\pi$: if $f \succ_\pi g \succ_\pi h$, then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)h \succ_\pi g \succ_\pi \beta f + (1-\beta)h$.

Axiom 4 (Nondegeneracy)**.** For all $\pi \in \Pi$, there exist $f, g \in \mathcal{F}_\pi$ such that $f \succ_\pi g$.

Axiom 5 (State Independence)**.** For all $\pi \in \Pi$, π-nonnull $E \in \sigma(\pi)$, $p, q \in \Delta X$, and $f \in \mathcal{F}_\pi$:

$$p \succsim_{\{S\}} q \iff \begin{pmatrix} p & E \\ f & E^\complement \end{pmatrix} \succsim_\pi \begin{pmatrix} q & E \\ f & E^\complement \end{pmatrix}.$$

State Independence has some additional content in our model. Not only is cardinal utility for consequences invariant to the event E in which it obtains, but also invariant to the decision maker's minimal level of awareness π.

These familiar axioms guarantee an Anscombe–Aumann expected utility representation for each $\succsim_\pi$: a probability measure $\mu_\pi : \sigma(\pi) \to [0,1]$ and an affine function $u : \Delta X \to \mathbb{R}$ such that $\int_S u \circ f \, d\mu_\pi$ is a utility representation of $\succsim_\pi$. Given a fixed partition π, the decision maker's preferences $\succsim_\pi$ are completely standard: she is probabilistically sophisticated on $\sigma(\pi)$ and evaluates lotteries linearly. Expected utility for a fixed level of awareness is not at odds with our model. The model's interest derives from the relationship between preferences across different partitions, i.e. in how the decision maker responds to updated awareness. The following axioms consider this relationship.

To consider an act f, the decision maker must be aware of the events which are necessary for its description, namely those in $\pi(f)$. If she was ignorant of $\pi(f)$, reading any description of f would immediately refine her understanding of the states. Similarly, when comparing two acts f and g, she must have the minimal awareness required to describe both f and g. This motivates the following binary relation $\succsim$ on $\mathcal{F}$.

Definition 5. For all $f, g \in \mathcal{F}$ define $f \succsim g$ if $f \succsim_{\pi(f,g)} g$.[5]

In words, the minimum awareness relation $\succsim$ reflects the decision maker's preference when presented with the coarsest possible descriptions of the two acts. The remaining axioms are defined on the global relation $\succsim$, without referencing the entire family of preferences. The minimum awareness relation $\succsim$ is theoretically powerful because it carries all the essential information about the indexed preferences $\{\succsim_\pi\}_{\pi\in\Pi}$. Suppose the analyst wanted to understand the decision maker's response to an act f which is expressed more finely than $\pi(f)$. Then the finer description π must entail some redundancies, for example, $f^{-1}(p) = E_1 \cup E_2$, but the description separately lists E_1 and E_2 even though they return the same lottery. There is an act very similar to f whose minimal expression does require separate expressions for E_1 and E_2: an act f' which assigns a very close but different lottery p' to E_2. Given the Anscombe–Aumann axioms, the decision maker's utility for the original act f under π is very similar to her utility for the nearby f' under $\pi = \pi(f')$.

The defined relation $\succsim$ is generally intransitive, since the frames $\pi(f,g)$, $\pi(g,h)$, and $\pi(f,h)$ required for pairwise comparisons of f, g, and h are generally distinct. One common relaxation of transitivity is acyclicity. A preference relation $\succsim$ is acyclic if its strict component $\succ$ does not admit any cycles. Given that $\succsim$ is complete, this is equivalent to the following definition.

Axiom 6* (Acyclicity)**.** For all acts $f_1, \ldots, f_n \in \mathcal{F}$,

$$f_1 \succ f_2, \ldots, f_{n-1} \succ f_n \Longrightarrow f_1 \succsim f_n.$$

[5] An alternative global relation is the comparison of certainty equivalents. Define $f \succsim^* g$ if $\mathrm{CE}(f) \succsim \mathrm{CE}(g)$ whenever $f \sim \mathrm{CE}(f) \in \Delta X$ and $g \sim \mathrm{CE}(g) \in \Delta X$. The examination of $\succsim^*$ is part of ongoing work.

It is well known that acyclicity of the preference relation is necessary and sufficient for its induced choice rule to be nonempty for any finite choice set. Since the presentation of an entire choice set of many acts will dramatically improve the decision maker's awareness, the interpretation in our setting is less direct. Here, the nonemptiness of the choice rule means that, for any finite set $A \subset \mathcal{F}$, we can assign some status quo act $f \in A$ such that if the decision maker is presented the minimal expression of any alternative $g \in A$, she will weakly prefer to keep f, $f \succsim g$.

Another justification of acyclicity is that it prevents the construction of Dutch book schemes which are strictly profitable at each trade. This justification is more tenuous in our interpretation of the model as one of awareness, because the preference relations that comprise the sequence in the hypothesis of Binary Bet Acyclicity are all indexed by distinct minimal partitions. If we present the decision maker with the first, the second, and then the third binary bets, she has become aware of more events than had she been presented the second and third bets in isolation. On the other hand, the preference notated in the axiom is really the latter. Perhaps a similar but more dexterously applied justification is that if there were n people who exhibit such behavior, we could construct a Dutch book between them by offering them separate choices. This could in principle be tested in a laboratory across subjects.

This discussion suggests a general issue with the observability of the model's primitives, which is related to issues with many dynamic models of preference. Once the decision maker's preferences under frame π are elicited, it is unclear how to elicit her preferences under coarser frames, since the awareness of subevents in π cannot be reversed. We think the best defense of the testability of our model, however, is that it embeds the standard Anscombe–Aumann model when one assumes that framing is irrelevant. Any model of decision making over acts, including the standard one, encounters these problems. One methodological choice is to assume away the thorny implications of framing for the theory and assume that a single relation without reference to framing is sufficient to capture decision making. This simplification is analogous to making an a priori econometric identification assumption to get a handle on the data. Another choice is not make this assumption, but then admit these problems up front as a limitation. Analogously, dropping identifying assumptions decreases statistical power. We are not arguing that making the first simplifying assumption is insensible, just that it finesses the problem away. These issues were always implicit in the standard model, but are given an explicit expression in our model.

While it might appear innocuous, Acyclicity is quite strong. Acyclicity precludes any meaningful notion of unawareness by forcing partition-independence.

Theorem 1. $\{\succsim_\pi\}_{\pi \in \Pi}$ *admits a partition-independent expected utility representation if and only if it satisfies the Ancsombe–Aumann axioms and Acyclicity.*[6]

Proof. See Appendix A.1.

[6]Theorem 1 remains true if Acyclicity is replaced with transitivity of $\succsim$. Also, recall the certainty equivalent relation $\succsim^*$ defined in Footnote 5: $f \succsim^* g$ if any certainty equivalent of f is weakly preferred to any certainty equivalent of g. Acyclicity of the minimum awareness relation $\succsim$ can also be replaced with monotonicity or weak admissibility of the certainty equivalent relation $\succsim^*$: $f \succsim^* g$ whenever $f(s) \succsim^* g(s)$ for all $s \in S$.

So, to allow for partition-dependent expected utility, Acyclicity must be further generalized. In particular, some types of cycles must be admitted. But, some structure is retained; arbitrary cycles are not allowed. We can specify exactly which cycles are allowed and prohibited, but first need to introduce some notation.

Definition 6. A sequence of events $E_1, E_2, \ldots$ is **sequentially disjoint** if $E_i \cap E_{i+1} = \emptyset$ for all i.

In particular, cycles involving simple binary bets across sequentially disjoint events are not allowed. Any cycle must be more complicated than simple comparisons of bets on disjoint events.

Axiom 6 (Binary Bet Acyclicity)**.** For any sequentially disjoint cycle of sets $E_1, \ldots, E_n, E_1$ and lotteries $p_1, \ldots, p_n; q \in \Delta X$,

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\mathsf{C}} \end{pmatrix} \succ \begin{pmatrix} p_2 & E_2 \\ q & E_2^{\mathsf{C}} \end{pmatrix}, \ldots, \begin{pmatrix} p_{n-1} & E_{n-1} \\ q & E_{n-1}^{\mathsf{C}} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_n^{\mathsf{C}} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\mathsf{C}} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\mathsf{C}} \end{pmatrix}.$$

Binary Bet Acylicity forces the decision maker to consistently evaluate simple likelihoods for disjoint events. If Binary Bet Acyclicity is satisfied, a money pump in the population must be more sophisticated than trading simple bets.

Once Acyclicity is relaxed to Binary Bet Acyclicity, the classic Sure-Thing Principle of Savage (1954) must be imposed to maintain a form of consistency.

Axiom 7 (Sure-Thing Principle)**.** For all events $E \subset S$ and acts $f, g, h, h; \in \mathcal{F}$,

$$\begin{pmatrix} f & E \\ h & E^{\mathsf{C}} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h & E^{\mathsf{C}} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} f & E \\ h' & E^{\mathsf{C}} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h' & E^{\mathsf{C}} \end{pmatrix}.$$

The standard justification for the Sure-Thing Principle is in establishing coherent conditional preferences. The evaluation of conditional probabilities for subevents on E should be independent on what happens on E^{C}. Because the expression of acts on E^{C} also confers some awareness, this axiom has additional content here. As discussed, comparing two acts requires awareness of certain events. When the range of h is disjoint from the ranges of f and g, the awareness needed to make the comparison in the left hand side can be divided into two parts: conditional awareness of subevents of E generated by f and g, and conditional awareness of subevents of E^{C} generated by h. The awareness needed to make the comparison in the right hand side can be similarly divided: conditional awareness of the *same* subevents of E generated by f and g, and awareness of possibly different subevents of E^{C} generated by h'. Since the conditional awareness required on E is similar for both comparisons and the acts being compared agree on E^{C}, the Sure-Thing Principle requires that the preferences are determined by where the acts differ on E.

The Sure-Thing Principle implies that the implicit subjective likelihood ratio of E to F can depend only on how finely E and F are described, and cannot depend on the expression of their complement $(E \cup F)^{\mathsf{C}}$. For example, a decision maker's assessment of the likelihood of a heart problem versus a skin problem might depend on how many kinds of lung problems are explicitly listed. Such a decision maker would violate the Sure-Thing Principle. This example suggests a background correlation in the likelihood of events; it is possible to have both heart and lung problems. Our intuition is that if the complete state space is correctly identified as a set of mutually exclusive events, so different combinations of joint heart and lung problems are separate states, then such examples are less worrisome.

We can now present the main representation result of the paper, characterizing partition-dependent expected utility:

Theorem 2. $\{\succsim_\pi\}_{\pi\in\Pi}$ *admits a partition-dependent expected utility representation if and only if it satisfies the Anscombe–Aumann axioms, Binary Bet Acyclicity, and the Sure-Thing Principle.*

Proof. See Appendix A.2

4.2 Uniqueness

While the utility function u over lotteries is unique up to positive affine transformations, the uniqueness of ν in the representation is surprisingly delicate. This delicacy also provides some intuition for Theorem 2. Our general strategy for identifying ν is to use an appropriate chain of partitions and betting preferences to calibrate the likelihood ratio $\nu(E)/\nu(F)$. For example, suppose $S = \{a, b, c\}$ and consider the ratio $\nu(\{a, b\})/\nu(\{a\})$. First, examine preferences indexed by the partition $\pi_1 = \{\{a, b\}; \{c\}\}$ to identify the likelihood ratio of $\{a, b\}$ to $\{c\}$. Next, the preferences indexed by $\pi_2 = \{\{a\}; \{b\}; \{c\}\}$ reveal the ratio of $\{c\}$ to $\{a\}$. The Sure-Thing Principle and Binary Bet Acyclicity suggest the following argument: the ratio of $\{a, b\}$ to $\{a\}$ is equal to the ratio of $\{a, b\}$ to $\{c\}$ times the ratio of $\{c\}$ to $\{a\}$, i.e. "the $\{c\}$'s cancel" and the revealed likelihood ratios multiply out. However, if $\{c\}$ is π_1-null, these ratios are undefined. Instead of achieving total uniqueness, the family of nonempty events segregates into equivalence classes of events which reach each other through sequentially disjoint chains of nonnull comparisons. Without further restrictions, ν is unique only up to scale transformations for all such equivalence classes. If all events are nonnull for all partitions, there are two equivalence classes, the universal event S and the collection $2^S \setminus \{S, \emptyset\}$ of all of its nonempty strict subsets. Then ν is identified on each up to a constant scalar multiple. This is the best we can hope for, since $\nu(S)$ is indeterminate because it matters only under the trivial partition $\pi = \{S\}$, in which case $\nu(S)/\nu(S) = 1$ no matter what value is assigned to $\nu(S)$. This motivates the following definition.

Axiom 8 (Event Reachability)**.** For any distinct nonnull events E and F different from S, there exists a sequentially disjoint sequence of nonnull events $E_1, \ldots, E_n$ such that $E = E_1$, $F = E_n$.

Event Reachability is immediately satisfied if all nonempty events are nonnull. Strict Admissibility is sometimes normatively invoked as a strong form of monotonicity or dominance.

Axiom 8* (Strict Admissibility)**.** If $f(s) \succsim g(s)$ for all $s \in S$ and $f(s') \succ g(s')$ for some $s \in S$, then $f \succ g$.

Strict Admissibility readily implies Event Reachability. Also, unlike in the standard Savage model, Strict Admissibility is not inconsistent with the desired representation, even if the state space is very rich. For example, if the set function is strictly positive, i.e. $\nu(E) > 0$ for any nonempty event E, then there are no nonempty null events and Strict Admissibility is satisfied. Since ν is not assumed to be additive, this will not conflict with its values for larger sets, as it would in the standard model. Such a set function suggests a decision maker who always put some small positive probability on any explicitly mentioned contingency, which seems behaviorally compelling.

The converse is false. As shown in the next example, Event Reachability is strictly weaker than Strict Admissibility.

Example 3 (Event Reachability $\not\Rightarrow$ Strict Admissibility)**.** Let $S = \{s_1, s_2, s_3\}$ and suppose that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation, where only the events $\{s_1\}$, $\{s_2\}$, $\{s_3\}$, $\{s_1, s_2\}$, and S have strictly positive ν-weight. The specified ν is nondegenerate. Strict Admissibility fails since some nonempty events are null. Event Reachability is satisfied: any two singletons are immediately comparable, and and a sequentially disjoint path from $\{s_1, s_2\}$ to either $\{s_1\}$ or $\{s_2\}$ can be constructed through $\{s_3\}$.

Of course, the value of ν will be indeterminate on the universal event S, which only appears as the unique cell of the trivial partition and divides by itself to equal unity, and on the empty event $\emptyset$, which is never assigned a consequence. Event Reachability is necessary and sufficient to determine the set function everywhere else up to a scalar multiple. This is the strongest identification possible, since this scalar multiple will always divide itself out.

Theorem 3. *Suppose that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation by (u, ν). The following are equivalent:*

(i) *$\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Event Reachability.*

(ii) *If (u', ν') also represents $\{\succsim_\pi\}_{\pi\in\Pi}$, then there exist numbers $a, c > 0$ and $b \in \mathbb{R}$ such that $u'(p) = au(p) + b$ for all $p \in \Delta X$ and $\nu'(E) = c\nu(E)$ for all $E \neq S$.*

Proof. See Appendix A.3

If the partition-dependent expected utility representation (u, ν) for $\{\succsim_\pi\}_{\pi\in\Pi}$ is uniquely determined in the sense of part (ii) of Theorem 3, we write that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a unique partition-dependent expected utility representation by (u, ν).

4.3 Monotonicity

We next identify a condition on preferences that corresponds to the restriction that ν is monotone with respect to set inclusion, or that $\nu(E) \leq \nu(F)$ whenever $E \subset F \subsetneq S$. While it seems natural to assume that the decision maker puts less weight on a subset of an event, experiments repeatedly detect otherwise. For example, Tversky and Kahneman (1983) document numerous examples of the conjunction fallacy, where subjects judge the intersection of different events to be strictly more likely than its components. When estimating the frequency of seven-letter words ending with "ing" versus seven-letter words with "n" as the sixth letter, subjects report a higher frequency for the former set, even though it is a strict subset of the latter. In addition, violations of monotonicity due to the representativeness heuristic, as famously demonstrated by the Linda problem, are remarkably robust despite "a series of increasingly desperate manipulations designed to induce subjects to obey the conjunction rule" (Tversky and Kahneman 1983, p. 299).[7] In our interpretation of unawareness, violations of monotonicity simply reflect a high degree of unawareness.

In general, we are agnostic about the monotonicity of ν. However, we will show later that monotonicity delivers several attractive features, hence deserves an explicit characterization. When the set function ν is unique up to a scalar multiple, as characterized in Theorem 3, the following condition guarantees that ν is monotone.

Axiom 9 (Monotonicity)**.** For all $E \subset F$ and $p, q, r, s \in \Delta X$ such that $p \succ q$,

$$s \succsim \begin{pmatrix} p & F \\ q & F^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} r & E \\ s & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & E \\ p & F \setminus E \\ q & F^{\complement} \end{pmatrix}.$$

Proposition 1. *Suppose $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a unique partition-dependent expected utility representation (u, ν). Then $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Monotonicity if and only if $E \subset F \subsetneq S$ implies $\nu(E) \leq \nu(F)$.*

Proof. See Appendix A.4.

Event Reachability is indispensable in Proposition 1. In general, there could exist one representation where ν is monotone, but another where ν' is not. Example 4 of Appendix A.4 demonstrates this explicitly. Without Event Reachability, we can only conclude that all subevents of null events remain null, i.e. if F is null and $E \subset F$, then E is also null.

The first interesting and potentially useful property of the Monotonicity axiom is that, once imposed, Event Reachability and Strict Admissibility are equivalent.

Proposition 2. *Suppose $|S| \geq 3$ and $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-dependent expected utility representation and satisfies Monotonicity. Then $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Event Reachability if and only if it satisfies Strict Admissibility.*

Proof. See Appendix A.5.

[7] In the Linda problem, subjects are told that "Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations." The subjects believe the event "Linda is a bank teller" is less probable than the event "Linda is a bank teller and is active in the feminist movement" (Tversky and Kahneman 1983, p. 297).

5 Measures of unawareness

5.1 Absolute and comparative correction for unawareness

Here, we introduce behavioral definitions and a quantitative measurement of a decision maker who incompletely corrects for her lack of awareness. We then suggest a method of comparing correction for unawareness across individuals using this quantitative measure.

We first review some formalities. Define the binary relation $\geq$ on Π by $\pi' \geq \pi$ if $\sigma(\pi') \supset \sigma(\pi)$, i.e. if π' is finer than π. In words, the decision maker is aware of more contingencies under π' than under π. While the relation is transitive, not all pairs of partitions can be ordered. This binary relation defines a lattice on Π, where the meet $\pi \wedge \pi'$ denotes the finest common coarsening of π and π' and the join $\pi \vee \pi'$ denotes the coarsest common refinement of π and π'. For any event E, let Π_E denote the set of all finite partitions of E.[8] Slightly abusing notation, if $E \subset S$ and $\pi'_E \in \Pi_E$, let $\pi \vee \pi'_E$ denote $\pi \vee [\pi'_E \cup \{E^{\complement}\}]$, which is the coarsest partition which includes all the cells in π and π'_E in its implied algebra. When it engenders no confusion, given a partition $\pi' \in \Pi$ and an event $E \in \sigma(\pi')$ of which the decision maker is aware under π', let $\pi'_E \in \Pi_E$ denote the restriction of π' to E: $\pi'_E = \{F \in \pi' : F \subset E\}$.

The following definitions of absolute under and overcorrection for unawareness do not depend on the particular utility representation forwarded in the previous section. The decision maker undercorrects for her unawareness of an event if she puts more relative likelihood on the event as she understands its contingencies better. Conversely, she overcorrects if she puts less likelihood on the event as she understands it better. We stress the correction for unawareness because the decision maker can try to adjust her assigned likelihood for the events which are not explicitly mentioned in the framing of the acts. In doing so, she may undershoot or overshoot the desired target. Is she happens to correct for her unawareness precisely, we cannot distinguish her behavior from that of someone who has full awareness.

Definition 7. Suppose $E \in \pi \in \Pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **undercorrects for unawareness of** π'_E **at** π if, for any $p, q, r \in \Delta X$ such that $q \succ r$:

$$\begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \succsim_\pi p \Longrightarrow \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \succsim_{\pi\vee\pi'_E} p$$

Suppose $\pi' \geq \pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **undercorrects for unawareness of** π' **at** π if $\{\succsim_\pi\}_{\pi\in\Pi}$ undercorrects for unawareness of π'_E for all $E \in \pi$.

Finally, $\{\succsim_\pi\}_{\pi\in\Pi}$ **undercorrects for unawareness** if $\{\succsim_\pi\}_{\pi\in\Pi}$ undercorrects for unawareness of π' for all $\pi' \geq \pi$.

In words, if the decision maker's certainty equivalent for a bet on the event E increases when she becomes aware of π'_E, then she is undercorrecting at the point when she is unaware of π'_E. One way to consider the definition is that she is willing to pay more to insure against contingency E

[8] We adopt the notational convention that $\Pi_\emptyset = \emptyset$.

as she becomes increasingly aware of its subevents. For example, violations of monotonicity entail severe undercorrections for unawareness.

Definition 8. Suppose $E \in \pi \in \Pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **overcorrects for unawareness of** π'_E **at** π if, for any $p, q, r \in \Delta X$ such that $q \succ r$:

$$p \succsim_\pi \begin{pmatrix} q & E \\ r & E^{\mathsf{C}} \end{pmatrix} \Longrightarrow p \succsim_{\pi\vee\pi'_E} \begin{pmatrix} q & E \\ r & E^{\mathsf{C}} \end{pmatrix}$$

Suppose $\pi' \geq \pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **overcorrects for unawareness of** π' **at** π if $\{\succsim_\pi\}_{\pi\in\Pi}$ overcorrects for unawareness of π'_E for all $E \in \pi$.

Finally, $\{\succsim_\pi\}_{\pi\in\Pi}$ **overcorrects for unawareness** if $\{\succsim_\pi\}_{\pi\in\Pi}$ overcorrects for unawareness of π' for all $\pi' \geq \pi$.

In an example given earlier, we considered a car owner who purchases too much warranty protection when she does not understand how her engine works. Such a consumer is overcorrecting to her unawareness.

When preferences admit a unique partition-dependent expected utility representation, subadditivity or superadditivity of the set function ν determines whether revealed likelihood increases or decreases as an event becomes more finely described.

Definition 9. A set function ν is **subadditive** if $\nu(E \cup F) \leq \nu(E) + \nu(F)$ whenever $E \cap F = \emptyset$ and $E \cup F \neq S$. A set function ν is **superadditive** if $\nu(E \cup F) \geq \nu(E) + \nu(F)$ whenever $E \cap F = \emptyset$ and $E \cup F \neq S$.

In the context of pure framing in support theory, Tversky and Koehler (1994) argue for and provide evidence suggesting subadditivity of the support function across disjunctions of hypotheses. Note that superadditivity is strictly weaker than convexity, $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$ for all $E, F \subset S$. Convexity is commonly assumed for capacities in Choquet integration or for value functions in cooperative games, but carries less behavioral content in terms of unawareness.

The disparity between beliefs across partitions can be quantitatively characterized by a ratio of the set function's value on the event E as she is aware and unaware of the subevents of π'_E.

Definition 10. Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Axioms 1–8, so is represented by a utility function u and a set function ν. If $E \neq S$ is nonnull, and $\pi'_E \in \Pi_E$, define the **coefficient of unawareness correction of** π'_E as

$$\lambda(\pi'_E) = \frac{\sum_{F\in\pi'_E} \nu(F)}{\nu(E)}.$$

The standard notion of risk aversion can be expressed behaviorally in terms of certainty equivalents, as a property of the utility function for wealth, or quantitatively through the Arrow–Pratt coefficient. Our proposed definition of under and overcorrection can be similarly tied to a structural condition on the set function ν, which can then be tied to a quantitative measure.

Proposition 3. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation (u,ν) and satisfies Strict Admissibility.*[9] *Then the following are equivalent:*

(i) $\{\succsim_\pi\}_{\pi\in\Pi}$ *undercorrects* [overcorrects] *for unawareness;*

(ii) ν *is subadditive* [superadditive]*;*

(iii) $\lambda(\pi'_E) \geq [\leq]\, 1$ *for all nonnull* $E \neq S$*,* $\pi'_E \in \Pi_E$*.*

We omit the straightforward proof of this result. We next introduce a relative notion of undercorrection for unawareness.

Definition 11. Suppose (u,ν) and (u',ν') uniquely represent $\{\succsim_\pi\}_{\pi\in\Pi}$ and $\{\succsim'_\pi\}_{\pi\in\Pi}$ in the sense of Theorem 3. Let λ,λ' denote their coefficients of unawareness correction. Then ν is **more underaware** than ν' if $\lambda(\pi_E) \geq \lambda'(\pi_E)$ for all $E \neq S$ with $\nu(E),\nu'(E) > 0$ and $\pi_E \in \Pi_E$.

An obvious deficiency in the comparative definition is its dependence on a component of partition-dependent expected utility representation. A more basic definition, without reference to a specific functional form, would be superior.[10] However, the concept seems like a reasonable one for the narrower space where our representation holds uniquely. The examples introduced earlier buttress the intuition.

First, recall Example 1: $\nu(E) = \varphi(\mu(E))$ for some finitely additive probability measure $\mu : 2^S \to [0,1]$ and increasing transformation $\varphi : [0,1] \to [0,1]$. Then $\{\succsim_\pi\}_{\pi\in\Pi}$ undercorrects for unawareness whenever φ is concave and overcorrects whenever φ is convex, so the absolute definition seems to work here. Moreover, suppose $(u,\varphi\circ\mu)$ and $(u',\varphi'\circ\mu)$ represent $\{\succsim_\pi\}_{\pi\in\Pi}$ and $\{\succsim'_\pi\}_{\pi\in\Pi}$. Then $\{\succsim_\pi\}_{\pi\in\Pi}$ is more underaware than $\{\succsim'_\pi\}_{\pi\in\Pi}$ whenever φ is a concave transformation of φ'.

Now consider Example 2, where $\nu(E) = \alpha + (1-\alpha)\mu(E)$ for some finitely additive probability measure μ and $\alpha \in [0,1]$. When $\alpha = 1$, ν is a constant set function and corresponds with the principle of insufficient reason which puts equal weight on all listed contingencies. Suppose $(u, \alpha\mu + (1-\alpha))$ and $(u', \beta\mu + (1-\beta))$ represent $\{\succsim_\pi\}_{\pi\in\Pi}$ and $\{\succsim'_\pi\}_{\pi\in\Pi}$. Then $\{\succsim_\pi\}_{\pi\in\Pi}$ is more underaware than $\{\succsim'_\pi\}_{\pi\in\Pi}$ if and only if $\alpha \geq \beta$. In words, a decision maker who is more biased towards the ignorance prior will exhibit more undercorrection for unawareness.

The comparative statics of Example 2 are actually a special case of a more general result. Suppose ν' is more underaware than ν. Some simple calculus verifies that $\alpha\nu + (1-\alpha)\nu'$ is more underaware than $\beta\nu + (1-\beta)\nu'$ if and only if $\alpha \geq \beta$. Therefore, the comparison of awareness is monotone across convex combinations in the desired manner.

5.2 From awareness of partitions to awareness of events

So far, we have formalized awareness with respect to partitions. One might be interested of the decision maker's awareness of specific events, independent of any partition of the state space. The

[9] Example 5 in Appendix A.6 shows that Strict Admissibility is indispensable for Proposition 3.

[10] We have an equivalent behavioral characterization of this definition which is independent of any particular utility representation, but at this point it is too complicated to be superior to this one.

literature on unawareness defines awareness either with respect to partitions or to specific events. The following is a notion of extreme unawareness for particular sets, and provides a more unified behavioral perspective of awareness for both partitions and for events. It starts with a primitive of preference with respect to partitions and identifies complete unawareness of events.

Definition 12. $\{\succsim_\pi\}_{\pi\in\Pi}$ is **completely unaware** of $E \subset S$ if E is nonnull and for all partitions $\{E, F, G\}$ of S and $p, q, r \in \Delta X$:

$$\begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \sim r \Longleftrightarrow \begin{pmatrix} p & F \\ q & E \cup G \end{pmatrix} \sim r.$$

In words, the decision maker never puts any weight on E unless it is explicitly described to her. In the first comparison, she attributes all the likelihood of receiving p to F because she is completely unaware of E; in the second comparison, all the likelihood of q is similarly attributed to G. Due to the framing of both acts, E remains occluded and the certainty equivalents are equal.

Definition 12 begins by distinguishing an event of which the decision maker is completely unaware from an event which the decision maker considers null. The following preference is not precluded by complete unawareness of E:

$$\begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \succ \begin{pmatrix} p' & E \\ p & F \\ q & G \end{pmatrix}.$$

Here, the presentation of the second act makes the decision maker aware of E, at which point she assigns it some positive likelihood. In contrast, this strict preference is precluded whenever E is null, because then the decision maker would be indifferent to whether p' or p was assigned to the impossible event E. Therefore, the primitives allow the analyst to distinguish unawareness and nullity from preferences over bets.

Proposition 4. *Suppose $|S| \geq 3$ and $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation (u, ν) where ν monotone. Then $\{\succsim_\pi\}_{\pi\in\Pi}$ is completely unaware of all nonempty events if and only if $\nu(E) = \nu(F)$ for all $E, F \neq \emptyset, S$.*

Proof. See Appendix A.7.

The extreme case of complete unawareness across all events is represented by a constant capacity where $\nu(E) = 1$ for every nonempty E. The decision maker places a uniform distribution over the events in her partition; extreme unawareness corresponds to the principle of insufficient reason. In fact, it is easily verified that the constant capacity is more underaware than any other monotone set function.

A Appendix

A.1 Proof of Theorem 1

For the necessity part, assume that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-independent expected utility representation (u,μ). Note that $f \succsim g$ if and only if $\int_S u\circ f\, d\mu \geq \int_S u \circ f\, d\mu$ for any $f,g\in\mathcal{F}$. Thus $\succsim$ is transitive, hence acyclic. The necessity of the Anscombe–Aumann axioms follows immediately from the standard Anscombe–Aumann Expected Utility Theorem.

We next prove the sufficiency part. The first five axioms provide a simple generalization of the Anscombe–Aumann Expected Utility Theorem.

Lemma 1. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the Anscombe–Aumann axioms. Then there exist an affine utility function $u:\Delta X\to\mathbb{R}$ with $[-1,1]\subset u(\Delta X)$ and a family of probability measures $\{\mu_\pi\}_{\pi\in\Pi}$ with $\mu_\pi:\sigma(\pi)\to[0,1]$ such that*

$$f\succsim_\pi g \Longleftrightarrow \int_S u\circ f\, d\mu_\pi \geq \int_S u\circ g\, d\mu_\pi.$$

Proof. For each $\pi\in\Pi$, Axioms 1–5 guarantee a probability measure μ_π on $(S,\sigma(\pi))$ and a non-constant affine vNM utility function $u_\pi:\Delta X\to\mathbb{R}$ such that $f\succsim_\pi g$ if and only if $\int_S u_\pi\circ f\, d\mu_\pi \geq \int_S u_\pi\circ g\, d\mu_\pi$, for all $f,g\in\mathcal{F}_\pi$. By State Independence, $p\succsim_\pi q$ if and only if $p\succsim_{\pi'} q$, therefore $u_\pi(p)\geq u_\pi(q)$ if and only if $u_{\pi'}(p)\geq u_{\pi'}(q)$. Then the uniqueness component of the standard Anscombe–Aumann Expected Utility Theorem implies that $u_{\pi'}$ is a positive affine transformation of u_π. By appropriately normalizing, we lose no generality by assuming $u_\pi = u_{\pi'} = u$. Nondegeneracy ensures that u is not constant, so we may further assume that its image contains the interval $[-1,1]$, again by appropriately normalizing. $\square$

Assume that $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the Anscombe-Aumann Axioms and Acyclicity. Let u and $\{\mu_\pi\}_{\pi\in\Pi}$ be as guaranteed by Lemma 1. We will first show that

$$\forall\pi\in\Pi\setminus\{\{S\}\} \text{ and } E\in\pi:\ \ \mu_\pi(E)=\mu_{\{E,E^\complement\}}(E) \tag{1}$$

Suppose for a contradiction that $\mu_\pi(E)>\mu_{\{E,E^\complement\}}(E)$ in (1). Let $\mu_\pi(E)>\alpha>\mu_{\{E,E^\complement\}}(E)$. Since the range of u contains the interval $[-1,1]$, there exist $p,q\in\Delta X$ such that $u(p)=1$ and $u(q)=0$. Define the act h by

$$h=\begin{pmatrix} p & E\\ q & E^\complement\end{pmatrix}.$$

Note that $\alpha p+(1-\alpha)q\succ h$. Let $f\in\mathcal{F}$ be such that $\pi(f)=\pi$ and for all $s\in S$, $u(f(s))<0$. Then there exists $\varepsilon\in(0,1)$ such that the act $h^\varepsilon\equiv(1-\varepsilon)h+\varepsilon f$ satisfies $\pi(h^\varepsilon)=\pi$ and $h^\varepsilon\succ_\pi \alpha p+(1-\alpha)q$. Then $h\succ h^\varepsilon\succ\alpha p+(1-\alpha)q\succ h$, a contradiction to $\succsim$ being acyclic. The argument for the case where $\mu_\pi(E)<\mu_{\{E,E^\complement\}}(E)$ is entirely symmetric, hence omitted.

Define $\mu: 2^S\to[0,1]$ by $\mu(\emptyset)\equiv 0$, $\mu(S)\equiv 1$, and $\mu(E)\equiv\mu_{\{E,E^\complement\}}(E)$ for $E\neq\emptyset, S$. To see that μ is finitely additive, let E,F be nonempty disjoint sets. If $E\cup F=S$, then $F=E^\complement$ so

$$\mu(E)+\mu(F)=\mu_{\{E,E^\complement\}}(E)+\mu_{\{E,E^\complement\}}(E^\complement)=1=\mu(E\cup F).$$

If $E\cup F\subsetneq S$, let $\pi=\{E,F,(E\cup F)^\complement\}$ and $\pi'=\{E\cup F,(E\cup F)^\complement\}$. Then by (1),

$$\mu(E)+\mu(F)=\mu_\pi(E)+\mu_\pi(F)=1-\mu_\pi((E\cup F)^\complement)=1-\mu_{\pi'}((E\cup F)^\complement)=\mu_{\pi'}(E\cup F)=\mu(E\cup F).$$

Therefore μ is a probability measure. To conclude, note that for any $\pi \in \Pi$, the definition of μ and (1) imply that $\mu_\pi(E) = \mu(E)$ for all $E \in \pi$. Hence (u, μ) is a partition-independent representation of $\{\succsim_\pi\}_{\pi\in\Pi}$.

A.2 Proof of Theorem 2

The necessity of the Anscombe–Aumann axioms follow from the standard Anscombe–Aumann Expected Utility Theorem. In the following two Lemmas, we check the necessity of the final two axioms.

Lemma 2. *If $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation, then $\succsim$ satisfies the Sure-Thing Principle.*

Proof. For any $f, g \in \mathcal{F}$, note that $D(f, g) \equiv \{s \in S : f(s) \neq g(s)\} \in \sigma(\pi(f, g))$, hence:

$$\begin{aligned}
f \succsim g \quad &\iff \quad f \succsim_{\pi(f,g)} g \\
&\iff \quad \int_{D(f,g)} u \circ f \, d\mu_{\pi(f,g)} \geq \int_{D(f,g)} u \circ g \, d\mu_{\pi(f,g)} \\
&\iff \quad \sum_{\substack{F \in \pi(f,g): \\ F \subset D(f,g)}} u(f(F))\nu(F) \geq \sum_{\substack{F \in \pi(f,g): \\ F \subset D(f,g)}} u(g(F))\nu(F),
\end{aligned}$$

where the second equivalence follows from multiplying both sides by $\sum_{F'\in\pi(f,g)} \nu(F')$.

Now, to demonstrate the Sure-Thing Principle, let $E \subset S$ and $f, g, h, h' \in \mathcal{F}$. Let

$$\begin{aligned}
\hat{f} &= \begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix}; & \hat{g} &= \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix}; \\
\hat{f}' &= \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix}; & \hat{g}' &= \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}.
\end{aligned}$$

Note that $D \equiv D(\hat{f}, \hat{g}) = D(\hat{f}', \hat{g}') \subset E$ and $\pi_D \equiv \{F \in \pi(\hat{f}, \hat{g}) : F \subset D(\hat{f}, \hat{g})\} = \{F \in \pi(\hat{f}', \hat{g}') : F \subset D(\hat{f}', \hat{g}')\}$. Hence by the observation made in the first paragraph:

$$\begin{aligned}
\hat{f} \succsim \hat{g} \quad &\iff \quad \sum_{F\in\pi_D} u(\hat{f}(F))\nu(F) \geq \sum_{F\in\pi_D} u(\hat{g}(F))\nu(F) \\
&\iff \quad \sum_{F\in\pi_D} u(f(F))\nu(F) \geq \sum_{F\in\pi_D} u(g(F))\nu(F) \\
&\iff \quad \sum_{F\in\pi_D} u(\hat{f}'(F))\nu(F) \geq \sum_{F\in\pi_D} u(\hat{g}'(F))\nu(F) \\
&\iff \quad \hat{f}' \succsim \hat{g}'.
\end{aligned}$$

□

Lemma 3. *If $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation, then $\succsim$ satisfies Binary Bet Acyclicity.*

Proof. First note that for any (possibly empty) disjoint events E and F, and (not necessarily distinct) lotteries $p, q, r \in \Delta X$, we have:

$$\begin{pmatrix} p & E \\ q & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & F \\ q & F^{\complement} \end{pmatrix} \qquad \iff \qquad [u(p) - u(q)]\nu(E) \geq [u(r) - u(q)]\nu(F).$$

To see the necessity of Binary Bet Acyclicity, let $E_1, \ldots E_n, E_1$ be a sequentially disjoint cycle of events and $p_1, p_2, \ldots, p_n; q \in \Delta X$ be such that

$$\forall i = 1, \ldots n-1: \quad \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}.$$

The observation made in the first paragraph implies that $[u(p_1) - u(q)]\nu(E_1) > [u(p_2) - u(q)]\nu(E_2) > \ldots > [u(p_n) - u(q)]\nu(E_n)$. Since $[u(p_1) - u(q)]\nu(E_1) > [u(p_n) - u(q)]\nu(E_n)$, we conclude that

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}. \qquad \square$$

We now move to proving the sufficiency of the axioms for the representation. Let u and $\{\mu_\pi\}_{\pi\in\Pi}$ be as guaranteed by Lemma 1. We now record two facts which rely only on the Anscombe–Aumann axioms and the Sure-Thing Principle.

Lemma 4. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the Ancsombe–Aumann axioms, hence admits a representation $(u, \{\mu_\pi\}_{\pi\in\Pi})$ as in Lemma 1. For any events E, F and partitions π, π':*

(i) *If $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Leftrightarrow \mu_{\pi'}(E) = 0$.*

(ii) *If $E, F \in \pi, \pi'$ and $E \cap F = \emptyset$, then $\mu_\pi(E)\mu_{\pi'}(F) = \mu_\pi(F)\mu_{\pi'}(E)$*

Proof. To prove part (i), it is enough to show that if $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Rightarrow \mu_{\pi'}(E) = 0$. Suppose that $\mu_\pi(E) = 0$. Select any two lotteries $p, q \in \Delta X$ satisfying $u(p) > u(q)$ and any two acts h, h' such that $\pi(h) = \pi$, and $\pi(h') = \pi'$. Then

$$\begin{pmatrix} p & E \\ h & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h & E^{\complement} \end{pmatrix}$$

by Lemma 1. Hence

$$\begin{pmatrix} p & E \\ h' & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h' & E^{\complement} \end{pmatrix}$$

by the Sure-Thing Principle. Since $u(p) > u(q)$, the last indifference can hold only if $\mu_{\pi'}(E) = 0$ by Lemma 1.

To prove part (ii), observe that if either side of the desired equality is zero, then part (ii) is immediately implied by part (i). So we may proceed assuming that both sides are strictly positive. Then all of the terms $\mu_\pi(E)$, $\mu_{\pi'}(F)$, $\mu_\pi(F)$, and $\mu_{\pi'}(E)$ are strictly positive. As before, select any two lotteries $p, q \in \Delta X$ such that $u(p) > u(q)$, and define a new lottery r by

$$r = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} p + \frac{\mu_\pi(F)}{\mu_\pi(E) + \mu_\pi(F)} q.$$

Select any two acts h, h' such that $p, q, r \notin h(S) \cup h'(S)$, $\pi(h) = \pi$, and $\pi(h') = \pi'$. By the choice of r and the expected utility representation of $\succsim_\pi$, we have:

$$\begin{pmatrix} p & E \\ q & F \\ h & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h & (E \cup F)^{\complement} \end{pmatrix}$$

Hence by the Sure-Thing Principle,

$$\begin{pmatrix} p & E \\ q & F \\ h' & (E\cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E\cup F \\ h' & (E\cup F)^{\complement} \end{pmatrix}.$$

This indifference, in conjunction with the expected utility representation of $\succsim_{\pi'}$, implies that

$$u(r) = \frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E)+\mu_{\pi'}(F)}u(p) + \frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E)+\mu_{\pi'}(F)}u(q).$$

We also have

$$u(r) = \frac{\mu_{\pi}(E)}{\mu_{\pi}(E)+\mu_{\pi}(F)}u(p) + \frac{\mu_{\pi}(F)}{\mu_{\pi}(E)+\mu_{\pi}(F)}u(q),$$

by the definition of r. Subtracting $u(q)$ from each side of the two expressions for $u(r)$ above, we obtain

$$\frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E)+\mu_{\pi'}(F)}[u(p)-u(q)] = \frac{\mu_{\pi}(E)}{\mu_{\pi}(E)+\mu_{\pi}(F)}[u(p)-u(q)],$$

which further simplifies to $\frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E)} = \frac{\mu_{\pi}(F)}{\mu_{\pi}(E)}$ since both sides of the previous equality are strictly positive. $\square$

By part (i) of Lemma 4, any event $E \in \pi, \pi'$ is π-null if and only if it is π'-null. Hence we can change quantifiers in the definitions of null and nonnull events. A nonempty event E is null if and only if E is π-null for *some* partition π with $E \in \pi$. Dually, an event E is nonnull if and only if E is π-nonnull for *every* partition π with $E \in \pi$.[11]

For any two disjoint nonnull events E, F, define the ratio:

$$\frac{E}{F} \equiv \frac{\mu_{\pi}(E)}{\mu_{\pi}(F)}$$

where π is a partition such that $E, F \in \pi$. The value of $\frac{E}{F}$ does not depend on the particular choice of π, by part (ii) of Lemma 4. Moreover, $\frac{E}{F}$ is well-defined and strictly positive since E and F are nonnull. Finally, $\frac{F}{E} \times \frac{E}{F} = 1$ by construction. The following appeals to Binary Bet Acyclicity in generalizing this equality.

Lemma 5. *For any sequentially disjoint cycle of nonnull events $E_1, \ldots, E_n, E_1$ and lotteries $p_1, p_2, \ldots, p_n; q \in \Delta X$ such that $u(q) = 0$ and $u(p_i) \in (0,1)$ for $i = 1, \ldots, n$:*

$$(\forall i = 1, \ldots n-1): \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Proof. It is enough to show that the hypothesis above implies

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Let $\bar{\varepsilon} \in (0,1)$ be such that $u(p_i) + \bar{\varepsilon} < 1$ for $i = 1, \ldots, n$. Since the range of the utility function u over

[11]Note that $\emptyset$ is null and S is nonnull by Nondegeneracy. Also, there may exist a nonnull event E, which is π-null for some π such that $E \in \sigma(\pi)$. From the above observation concerning the quantifiers, this can only be possible if E is not a cell in π but a union of its cells. This would correspond to a representation where, for example, E is a disjoint union of two subevents $E = E_1 \cup E_2$ and $\nu(E) > 0$, yet $\nu(E_1) = \nu(E_2) = 0$.

lotteries contains the unit interval $[-1,1]$, for each $\varepsilon \in (0,\bar{\varepsilon})$ and $i \in \{1,\ldots,n\}$, there exists $p_i(\varepsilon) \in \Delta X$ such that $u(p_i(\varepsilon)) = u(p_i) + \varepsilon^i$, where ε^i refers to the ith power of ε.[12] The expected utility representation of Lemma 1 and the fact that E_i is nonnull implies that for sufficiently small $\varepsilon \in (0,\bar{\varepsilon})$,

$$\begin{pmatrix} p_i(\varepsilon) & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1}(\varepsilon) & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix},$$

for $i = 1,\ldots,n-1$. By Binary Bet Acyclicity, this implies

$$\begin{pmatrix} p_1(\varepsilon) & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n(\varepsilon) & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Appealing to the continuity of the expected utility representation of Lemma 1 in the assigned lotteries $f(s)$ and taking $\varepsilon \to 0$ proves the desired conclusion. □

Lemma 6. *For any sequentially disjoint cycle of nonnull events $E_1,\ldots,E_n,E_1$:*

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \cdots \times \frac{E_{n-1}}{E_n} \times \frac{E_n}{E_1} = 1.$$

Proof. The case where $n = 2$ immediately follows from our definition of event ratios, so assume that $n \geq 3$. Fix $t_1 > 0$, and recursively define

$$t_i = t_1 \times \frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \ldots \times \frac{E_{i-1}}{E_i}.$$

for $i = 2,\ldots,n$. By selecting a sufficiently small t_1, we may assume that $t_1, \ldots t_n \in (0,1)$. Also note that $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$ for $i = 1,\ldots,n-1$. Recall the range of the utility function u over lotteries contains the unit interval $[-1,1]$, so there exist lotteries $p_1,\ldots,p_n,q \in \Delta X$ such that $u(p_i) = t_i$ for $i = 1,\ldots,n$ and $u(q) = 0$.

Fix any $i \in \{1,\ldots,n-1\}$. Let $\pi = \{E_i, E_{i+1}, (E_i \cup E_{i+1})^{\complement}\}$. Since $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$, we have $\mu_\pi(E_{i+1})u(p_{i+1}) = \mu_\pi(E_i)u(p_i)$. Hence:

$$\begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}$$

by the expected utility representation of Lemma 1. Since the above indifference holds for any $i \in \{1,\ldots,n-1\}$, by Lemma 5, we have

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Hence by the expected utility representation of $\succsim_\pi$ for $\pi = \{E_1, E_n, (E_1 \cup E_n)^{\complement}\}$, we have $\mu_\pi(E_1)u(p_1) = \mu_\pi(E_n)u(p_n)$. This implies $\frac{t_n}{t_1} = \frac{E_1}{E_n}$. Recalling the construction of t_n, we then have the desired conclusion:

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \ldots \times \frac{E_{n-1}}{E_n} = \frac{E_1}{E_n}.$$ □

We can now conclude the proof of sufficiency. Let $\mathcal{E}$ denote the collection of all nonnull events, which is nonempty since Nondegeneracy ensures $S \in \mathcal{E}$. Define the binary relation $\approx$ on $\mathcal{E}$ by $E \approx F$ (we read it as F is reachable from E) if there exist a sequentially disjoint sequence of nonnull events $E_1,\ldots,E_n$ with

[12] We invoke the Axiom of Choice by assuming that we can fix a lottery $p_i(\varepsilon)$ for each $\varepsilon \in (0,\bar{\varepsilon})$. Lemma 5 remains true without invoking the Axiom of Choice, but requires a longer proof.

$E = E_1$ and $F = E_n$. The relation $\approx$ is reflexive, symmetric, and transitive, defining an equivalence relation on $\mathcal{E}$. For any nonnull $E \in \mathcal{E}$, let $[E] = \{F \in \mathcal{E} : E \approx F\}$ denote the equivalence class of E with respect to $\approx$ (the reach of E). Let $\mathcal{E}/\approx = \{[E] : E \in \mathcal{E}\}$ denote the quotient set of all equivalence classes of $\mathcal{E}$ modulo $\approx$, with a generic class $R \in \mathcal{E}/\approx$.[13] Select a representative event $G_R \in R$ for every equivalence class $R \in \mathcal{E}/\approx$, invoking the Axiom of Choice if the quotient is uncountable.

We next define ν. For all null E, let $\nu(E) = 0$. For every class $R \in \mathcal{E}/\approx$, *arbitrarily assign* a positive value $\nu(G_R) > 0$ for its representative. We will conclude by defining $\nu(E)$, for any $E \in \mathcal{E} \setminus \{S\}$. If $E = G_{[E]}$, then E represents its equivalence class and $\nu(E)$ has been assigned. Otherwise, whenever $E \neq G_{[E]}$, since $E \approx G_{[E]}$, there exists a sequentially disjoint path of nonnull events $E_1, \dots, E_n$ such that $E = E_1$, $G_{[E]} = E_n$. Then let:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Note that the definition of $\nu(E)$ above is independent of the particular choice of the path $E_1, \dots, E_n$, because for any other such sequentially disjoint path of nonnull events $E = F_1, \dots, F_m = G_{[E]}$:

$$\frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \frac{F_m}{F_{m-1}} \times \ldots \times \frac{F_2}{F_1} = 1$$

by Lemma 6.

We will next verify that $\nu \colon 2^S \setminus \{S\} \to \mathbb{R}_+$ defined above is a nondegenerate set function satisfying

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)} \tag{2}$$

for any event $E \in \pi$ of any partition $\pi \in \Pi \setminus \{\{S\}\}$.

Let $\pi \in \Pi \setminus \{\{S\}\}$. By Nondegeneracy and the expected utility representation for $\succsim_\pi$, there exists a π-nonnull $F \in \pi$. Then, since Lemma 4 implies π-nonnull events are nonnull, F is nonnull so the denominator on the right hand side of Equation (2) is strictly positive, so the fraction is well-defined. This also implies that ν is a nondegenerate set function. Observe that Equation (2) immediately holds if E is null, since then $\nu(E) = 0$ and $\mu_\pi(E) = 0$ follows from E being π-null. Let $\mathcal{E}_\pi \subset \pi$ denote the nonnull cells of π. To finish the proof of the Theorem, we will show that $\frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{\nu(E)}{\nu(F)}$ for any distinct $E, F \in \mathcal{E}_\pi$. Along with the fact that $\sum_{E \in \mathcal{E}_\pi} \mu_\pi(E) = 1$, this will prove Equation (2).

Let $E, F \in \mathcal{E}_\pi$ be distinct. Note that $[E] = [F]$ since E and F are disjoint. Suppose first that neither E nor F is $G_{[E]}$. Then there exist a sequentially disjoint path of nonnull events $E_1, \dots, E_n$ such that $E = E_1$, $G_{[E]} = E_n$, and:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

But then $F, E_1, \dots, E_n = G_{[E]}$ forms such a path from F to $G_{[E]}$, hence we have:

$$\nu(F) = \frac{F}{E_1} \times \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Dividing the term for $\nu(E)$ by the term for $\nu(F)$, we obtain $\frac{E}{F} = \frac{\nu(E)}{\nu(F)}$.

The other possibility is that exactly one of E or F (without loss of generality E) is $G_{[E]}$. Then the

[13] Note that $[S] = \{S\}$ and $E \approx F$ for any disjoint nonnull E, F.

nonnull events $F = E_1, E_2 = E$, make up a path from F to $E = G_{[E]}$. Then

$$\nu(F) = \frac{F}{E} \times \nu(E)$$

as desired.

A.3 Proof of Theorem 3

We maintain the notation and the results established in the proof of Theorem 2 in Appendix A.2. We first show the "(i) $\Rightarrow$ (ii)" part. Suppose that (u, ν) and (u', ν') are partition-dependent expected utility representation of $\{\succsim_\pi\}_{\pi \in \Pi}$ and that Event Reachability is satisfied. For each $\pi \in \Pi$, let μ_π and μ'_π respectively denote the probability distributions derived from ν and ν' by Equation (2). Applying the uniqueness component of the Anscombe-Aumann Expected Utility Theorem to $\succsim_\pi$, we have $\mu_\pi = \mu'_\pi$ and $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.

If $E \neq S$ is a null event, then $\nu(E) = \mu_\pi(E) = 0 = \mu'_\pi(E) = \nu'(E)$ for any partition π with $E \in \pi$. Also note that if E, F are two disjoint nonnull events, then

$$\frac{\nu(E)}{\nu(F)} = \frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{E}{F} = \frac{\mu'_\pi(E)}{\mu'_\pi(F)} = \frac{\nu'(E)}{\nu'(F)}.$$

We will next extend the equality $\frac{\nu(E)}{\nu(F)} = \frac{\nu'(E)}{\nu'(F)}$ to any pair of distinct (but not necessarily disjoint) nonnull events E and F different from S, in order to conclude that there exists for $c > 0$ such that $\nu'(E) = c\nu(E)$ for all $E \neq S$. Let E and F be two distinct nonnull events different from S. By Event Reachability, there exist nonnull events $E_1, \ldots, E_n$ such that $E = E_1$, $F = E_n$, $E_i \cap E_{i+1} = \emptyset$ for $i = 1, \ldots, n-1$. Then:

$$\frac{\nu(E)}{\nu(F)} = \frac{\nu(E_1)}{\nu(E_2)} \times \ldots \times \frac{\nu(E_{n-1})}{\nu(E_n)} = \frac{\nu'(E_1)}{\nu'(E_2)} \times \ldots \times \frac{\nu'(E_{n-1})}{\nu'(E_n)} = \frac{\nu'(E)}{\nu'(F)}$$

where the second equality follows from E_i and E_{i+1} being disjoint for $i = 1, \ldots, n-1$. Thus ν' is a scalar multiple of ν, determined by the constant $c = \nu(E)/\nu'(E)$ for any nonnull set E.

To see the "(i) $\Leftarrow$ (ii)" part, suppose that Event Reachability is not satisfied. Then the reachability relation $\approx$ defined in the proof of Theorem 2 has at least two different equivalence classes R and R' other than $\{S\}$. It is then possible to construct two partition-dependent expected utility representations (u, ν) and (u, ν') of $\{\succsim_\pi\}_{\pi \in \Pi}$ such that $\nu(G_R) = \nu'(G_R) > 0$ and $\nu(G_{R'}) = 2\nu'(G_{R'}) > 0$. Note that there does not exist a $c > 0$ such that $\nu(E) = \nu'(E)$ for all $E \neq S$.

A.4 Proof of Proposition 1

Suppose $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a (not necessarily unique) partition-dependent expected utility representation (u, ν). Then for any events $E \subset F$ and $p, q, r, s \in \Delta X$:

$$s \succsim \begin{pmatrix} p & F \\ q & F^{\mathsf{C}} \end{pmatrix} \quad \Longleftrightarrow \quad u(s)[\nu(F) + \nu(F^{\mathsf{C}})] \geq u(p)\nu(F) + u(q)\nu(F^{\mathsf{C}})$$

$$\begin{pmatrix} r & E \\ s & E^{\mathsf{C}} \end{pmatrix} \succsim \begin{pmatrix} r & E \\ p & F \setminus E \\ q & F^{\mathsf{C}} \end{pmatrix} \quad \Longleftrightarrow \quad u(s)[\nu(F \setminus E) + \nu(F^{\mathsf{C}})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\mathsf{C}}).$$

The next Lemma shows that the existence of a partition-dependent expected utility representation with a monotone set function ν implies Monotonicity. This is true even without Event Reachability, or without the uniqueness of ν, providing a stronger version of the necessity of the axiom required in Proposition 1.

Lemma 7. *If $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a (not necessarily unique) partition-dependent expected utility representation by (u,ν) and ν is monotone, then $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Monotonicity.*

Proof. Let $E \subset F$ and $p,q,r,s \in \Delta X$ such that $p \succ q$ and $u(s)[\nu(F)+\nu(F^{\complement})] \geq u(p)\nu(F)+u(q)\nu(F^{\complement})$. If $F = S$ or if $\nu(F\backslash E)+\nu(F^{\complement}) = 0$, then the desired conclusion holds. Otherwise $F \subsetneq S$ and $\nu(F\backslash E)+\nu(F^{\complement}) > 0$ and $\nu(F)+\nu(F^{\complement}) > 0$ by nondegeneracy of ν. Since $F \setminus E \subset F \subsetneq S$, by monotonicity of ν we have $\nu(F) \geq \nu(F\setminus E)$. Hence

$$\frac{\nu(F)}{\nu(F)+\nu(F^{\complement})} \geq \frac{\nu(F\setminus E)}{\nu(F\setminus E)+\nu(F^{\complement})}.$$

But then since $u(p) > u(q)$, the inequality:

$$u(s) \geq \frac{\nu(F)}{\nu(F)+\nu(F^{\complement})}u(p) + \frac{\nu(F^{\complement})}{\nu(F)+\nu(F^{\complement})}u(q)$$

implies

$$u(s) \geq \frac{\nu(F\setminus E)}{\nu(F\setminus E)+\nu(F^{\complement})}u(p) + \frac{\nu(F^{\complement})}{\nu(F\setminus E)+\nu(F^{\complement})}u(q).$$

Therefore $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Monotonicity. $\square$

The next Example shows that, without Event Reachability, we cannot guarantee monotonicity of ν for *every* partition-dependent expected utility representation (u,ν) of $\{\succsim_\pi\}_{\pi\in\Pi}$. It requires a state space with at least three elements; otherwise, any ν is trivially monotone according to our definition.

Example 4. Consider an arbitrary state space S with $|S| \geq 3$, and fix an nonempty event $A \subsetneq S$. Define ν by

$$\nu(E) = \begin{cases} 1 & \text{if } E \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for any event $E \neq S$. Note that ν is nondegenerate. Let $\{\succsim_\pi\}_{\pi\in\Pi}$ be represented by (u,ν) for some non-constant u. Any event B such that $A \subset B \subsetneq S$ has nonempty intersection with all nonnull events, hence it can not be linked to any other nonnull set through sequentially disjoint nonnull sets: In the notation of the proof of Theorem 2, such an event B's reachability class $[B]$ consists of only B. Hence although ν itself is monotone, it is straightforward to verify that ν' obtained from ν by changing $\nu(B)$ to $\frac{1}{2}$ continues to represent the same preference. Moreover if we choose B such that $|B| \geq 2$, then there exists a C such that $C \subsetneq B$ and $\nu'(C) = 1 > \frac{1}{2}\nu'(B)$, so ν' is not monotone.

We show in the next lemma that it is possible to guarantee a weaker version of monotonicity of the set function from the Monotonicity of $\{\succsim_\pi\}_{\pi\in\Pi}$: subsets of null events should also be null.

Lemma 8. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a (not necessarily unique) partition-dependent expected utility representation by (u,ν). If $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Monotonicity, then:*

$$E \subset F \;\&\; \nu(F) = 0 \;\Rightarrow\; \nu(F\setminus E) = 0.$$

Proof. Suppose that there exist events E, F such that $E \subset F$ and $\nu(F \setminus E) > \nu(F) = 0$. Since $\nu(F) = 0$, by non-degeneracy of ν, we have $\nu(F^{\complement}) > 0$. Let $p, q, s \in \Delta X$ be such that $u(p) > u(q) = u(s)$. Then

$$u(s)[\nu(F) + \nu(F^{\complement})] = u(p)\nu(F) + u(q)\nu(F^{\complement}),$$

so by Monotonicity, we should have

$$u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement}).$$

However the latter inequality is not possible, since $u(p) > u(q) = u(s)$ and $\nu(F \setminus E) > 0$, a contradiction. $\square$

In the next Lemma, we prove the sufficiency of the Monotonicity axiom for the existence of a monotone representation in Proposition 1, given Event Reachability and the uniqueness of the set function up to scalar multiples.

Lemma 9. *Suppose $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a unique partition-dependent expected utility representation by (u, ν). If $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Monotonicity, then ν is monotone.*

Proof. We prove by contraposition. Suppose that ν is not monotone. Then there exist events E, F such that $E \subset F \subsetneq S$ and $\nu(F \setminus E) > \nu(F)$. By Lemma 8, we can assume that $\nu(F) > 0$. We also have that $\nu(F^{\complement}) > 0$, because otherwise by Lemma 8, any subevent of $F^{\complement}$ is null, hence F and $F \setminus E$ are nonnull events that can not be linked by sequentially disjoint nonnull events, contradicting Event Reachability.

Let $p, q, s \in \Delta X$ be such that $u(p) > u(q)$ and

$$s = \frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} p + \frac{\nu(F^{\complement})}{\nu(F) + \nu(F^{\complement})} q.$$

Then $u(s)[\nu(F) + \nu(F^{\complement})] = u(p)\nu(F) + u(q)\nu(F^{\complement})$, so by Monotonicity, we have $u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement})$. Together with $u(p) > u(q)$, these imply:

$$\frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} \geq \frac{\nu(F \setminus E)}{\nu(F \setminus E) + \nu(F^{\complement})},$$

a contradiction to $\nu(F \setminus E) > \nu(F)$ and $\nu(F^{\complement}) > 0$. $\square$

A.5 Proof of Proposition 2

Given the existence of a partition-dependent expected utility representation, Strict Admissibility is equivalent to all nonempty events being nonnull. The "if" part is immediate. We proceed contrapositively to prove the "only if" part. Let $\{\succsim_\pi\}_{\pi \in \Pi}$ be represented by (u, ν). Now suppose that there is a nonempty null event E. By nondegeneracy of ν, $E^{\complement}$ is nonnull. By Lemma 8, all subevents of E are null. If there is an event B such that $E^{\complement} \subset B \subsetneq S$, then B is nonnull by Lemma 8. Hence $E^{\complement}$ and B are two nonnull events that are not linked by sequentially disjoint nonnull sets, so Event Reachability fails. If there is no such event B, then since $|S| \geq 3$, $E^{\complement}$ must consist of at least two elements. In this case, let $E^{\complement} = E_1 \cup E_2$, where E_1 and E_2 are nonempty and disjoint. Then $\{E_1, E_2, E\}$ is a partition of S where E is null, so one of the other two events, say E_i, is nonnull by nondegeneracy of ν. But then $E^{\complement}$ and E_i are two nonnull events that are not linked by sequentially disjoint nonnull sets, so again Event Reachability fails.

A.6 Indispensability of Strict Admissibility in Proposition 3

Example 5. Let $S = \{s_1, s_2, s_3, s_4\}$ and suppose that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation (u, ν), where $\nu(\{s\}) = 1$ for all $s \in S$, $\nu(\{s_1, s_2\}) = \nu(\{s_1, s_3\}) = \nu(\{s_2, s_3\}) = 3$, and $\nu(E) = 0$ for any other event $E \neq S$. Strict Admissibility fails since some nonempty events are null. It can be verified that ν is nondegenerate, Event Reachability is satisfied, and $\{\succsim_\pi\}_{\pi\in\Pi}$ undercorrects for unawareness. However, ν is not subadditive since $\nu(\{s_1, s_2\}) > \nu(\{s_1\}) + \nu(\{s_2\})$ and $\lambda(\{\{s_1\}, \{s_2\}\}) = \frac{2}{3} < 1$.

A.7 Proof of Proposition 4

It is easily seen that the "if" part of Proposition 4 holds even without imposing monotonicity of ν. To see that monotonicity is indispensable for the "only if" part to hold, consider the following example.

Example 6. Let $S = \{s_1, s_2, s_3\}$ and suppose that $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation (u, ν), where $\nu(\{s_1\}) = \nu(\{s_2, s_3\}) = 1$, $\nu(\{s_2\}) = \nu(\{s_1, s_3\}) = 2$, and $\nu(\{s_3\}) = \nu(\{s_1, s_2\}) = 3$. Strict Admissibility is satisfied therefore the partition-dependent expected utility representation is unique. The set function ν is not monotone since $\nu(\{s_2\}) > \nu(\{s_2, s_3\})$. It is easy to see that $\{\succsim_\pi\}_{\pi\in\Pi}$ is completely unaware of all nonempty events, however ν is not constant on nonempty and nonuniversal events.

We now prove the "only if" part. Note that Strict Admissibility is satisfied since complete unawareness of an event requires it to be nonnull. We first show that

$$\frac{\nu(E \cup F)}{\nu(G)} = \frac{\nu(F)}{\nu(E \cup G)} \tag{3}$$

for any three element partition $\{E, F, G\}$ of S. The fractions above are well defined, since strict admissibility guarantees that the denominators do not vanish. To see (3), let $p, q, r \in \Delta X$ be such that $u(p) > u(q)$ and

$$\frac{\nu(E \cup F)}{\nu(E \cup F) + \nu(G)} u(p) + \frac{\nu(G)}{\nu(E \cup F) + \nu(G)} u(q) = u(r) \iff \begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \sim r. \tag{4}$$

By complete unawareness of E, we have

$$\frac{\nu(F)}{\nu(F) + \nu(E \cup G)} u(p) + \frac{\nu(E \cup G)}{\nu(F) + \nu(E \cup G)} u(q) = u(r) \iff \begin{pmatrix} p & F \\ q & E \cup G \end{pmatrix} \sim r. \tag{5}$$

Since $u(p) > u(q)$, (4) and (5) imply that

$$\frac{\nu(E \cup F)}{\nu(E \cup F) + \nu(G)} = \frac{\nu(F)}{\nu(F) + \nu(E \cup G)}$$

which is equivalent to (3).

We next show that $\nu(A) = \nu(A^{\complement})$ if $A \neq \emptyset, S$. To see this, note that since there are at least three states A or $A^{\complement}$ is not a singleton. Without loss of generality suppose that A has at least two elements and let $\{A_1, A_2\}$ be a two element partition of A, then

$$\frac{\nu(A)}{\nu(A^{\complement})} = \frac{\nu(A_1 \cup A_2)}{\nu(A^{\complement})} = \frac{\nu(A_2)}{\nu(A_1 \cup A^{\complement})} == \frac{\nu(A_2 \cup A^{\complement})}{\nu(A_1)} = \frac{\nu(A^{\complement})}{\nu(A_1 \cup A_2)} = \frac{\nu(A^{\complement})}{\nu(A)},$$

by iterated application of (3), hence $\nu(A) = \nu(A^{\complement})$ as desired.

Take any distinct events $E, F \neq \emptyset, S$. If $E \setminus F \neq \emptyset$ then

$$\nu(E \setminus F) \leq \nu(E) = \nu(E^{\complement}) \leq \nu((E \setminus F)^{\complement}) = \nu(E \setminus F)$$

where the inequalities follow from monotonicity of ν, hence $\nu(E) = \nu(E \setminus F)$. Similarly

$$\nu(E \setminus F) \leq \nu(F^{\complement}) = \nu(F) \leq \nu((E \setminus F)^{\complement}) = \nu(E \setminus F),$$

hence $\nu(F) = \nu(E \setminus F) = \nu(E)$ as desired. The case where $F \setminus E \neq \emptyset$ is symmetric, therefore omitted.

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