Appendix V
The kinked and smooth specifications

[1] The kinked specification

The kinked utility function is so-called because the indifference curves have a "kink" at all portfolios where $x_1 = x_3$. The parametric specification we use has the form

$$U(\mathbf{x}; \alpha, \rho) = -\frac{2}{3}\alpha \exp\{-\rho x_{\min}\} - \frac{1}{3}\exp\{-\rho x_2\} - \frac{2}{3}(1-\alpha)\exp\{-\rho x_{\max}\}, \tag{1}$$

where $0 \geq \alpha \geq 1$ is the ambiguity parameter and ρ is the coefficient of risk aversion. The distinguishing feature of this specification is its dependence on the minimum and maximum payoffs, $x_{\min} = \min\{x_1, x_3\}$ and $x_{\max} = \max\{x_1, x_3\}$, between the two ambiguous states, 1 and 3. The agent knows that the probabilities of states 1 and 3 lie between 0 and $\frac{2}{3}$. In the best case scenario, the probability of the state in which he receives $x_{\max}$ is $\frac{2}{3}$; in the worst case scenario, it is zero. What equation 1 says is that the agents's utility is a weighted average, with weights α and $1-\alpha$, of the expected utility in the worst-case and best-case scenarios: $\frac{1}{2} < \alpha \leq 1$ indicate preferences that are ambiguity averse, $0 \leq \alpha < \frac{1}{2}$ indicate preferences that are ambiguity seeking, and if $\alpha = \frac{1}{2}$ we have the standard SEU representation.

We next demonstrate how this kinked functional form can be generated by different classes of preferences.

[1.1] Maxmin Expected Utility with flexible priors

The Maxmin Expected Utility (MEU) model of Gilboa and Schmeidler (1989) evaluates a portfolio by its minimal expected utility over a set of subjective prior beliefs. This minimization over a non-singleton set can be interpreted as aversion to ambiguity. The general form of the MEU model is

$$U(\mathbf{x}) = \min_{\pi \in \Pi} \int_S u(x_s)\, d\pi(s),$$

where $\Pi \subseteq \Delta S$ is a closed convex set of prior beliefs over states.

Connecting the general MEU model to our kinked specification assumes that the utility over tokens takes the CARA form and that the set of priors is symmetric about $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. In particular, the set of priors is

$$\Pi_\theta = \left\{\pi : \pi_2 = \tfrac{1}{3},\ \tfrac{1}{3} - \theta \leq \pi_1 \leq \tfrac{1}{3} + \theta,\ \pi_3 = \tfrac{2}{3} - \pi_1\right\}$$

for some $0 \leq \theta \leq \frac{1}{3}$. Larger values of θ indicate a larger set of priors, hence more ambiguity. This reduces the general MEU model to the following two-parameter formula:

$$U(\mathbf{x}; \delta, \rho) = -\left(\tfrac{1}{3} + \delta\right)\exp\{-\rho x_{\min}\} - \tfrac{1}{3}\exp\{-\rho x_2\} - \left(\tfrac{1}{3} - \delta\right)\exp\{-\rho x_{\max}\}.$$

This equation is exactly equation 1 with a change of variables, letting $\alpha = \frac{1}{2} + \frac{3}{2}\theta$.

[1.2] Choquet Expected Utility with flexible capacity

The Choquet Expected Utility (CEU) model of Schmeidler (1989) is related to MEU and takes the following general form:

$$U(\mathbf{x}) = \int_S u(x_s)\, d\nu(s),$$

where ν is a nonadditive capacity over the state space. Ambiguity in the CEU model is captured by the convexity of the capacity ν.[1],[2]

Any CEU representation with a convex capacity can be rewritten as an MEU representation where the set of priors is the core of the capacity. Correspondingly, if we assume CARA utility over tokens and that the capacity is symmetric over the two ambiguous states, then the CEU model reduces to the parameterized MEU model with symmetric priors presented in the previous section. In particular, if the capacity obeys:

$$\begin{aligned} \nu(\{1\}) = \nu(\{3\}) &= \tfrac{1}{3} - \theta, \qquad \nu(\{2\}) = \tfrac{1}{3}, \\ \nu(\{1,2\}) = \nu(\{2,3\}) &= \tfrac{2}{3} - \theta, \qquad \nu(\{1,3\}) = \tfrac{2}{3}, \end{aligned}$$

for some $0 \leq \theta \leq \frac{1}{3}$, then the implied Choquet integral reduces to equation 1, via the same change of variables $\alpha = \frac{1}{2} + \frac{3}{2}\theta$.

[1.3] Contraction Expected Utility with fixed information

The contraction model of Gajdos et al. (2008) incorporates objective information about the set of possible prior distributions over states. It enriches the standard subjective setup by considering acts or portfolios paired with some set of objectively known possible priors. The agent partially contracts this set towards its center and then applies the MEU criterion to this smaller set of priors. The general representation is

$$U(\mathbf{x}) = \min \left\{ \int_S u(x_s)\, d\pi(s) : \pi \in (1-\epsilon)\{s(\Pi)\} + \epsilon \Pi \right\},$$

where $s(\Pi) \in \Delta S$ is the Steiner point (a geometric notion of the center) of the set Π of objectively specified priors.[3] Larger values of $\epsilon \in [0,1]$ place more weight on the entire set of possible priors Π and, hence, suggest more ambiguity.

The experimental choice problem can be represented in this form, where every portfolio is paired with the same set of objective priors, namely $\Pi = \{\pi : \pi_2 = \frac{1}{3} \text{ and } \pi_1 + \pi_3 = \frac{2}{3}\}$. Its Steiner point is $s(\Pi) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. As Hayashi and Wada (2010) mention, the contraction model with a fixed set of possible priors is identical to a special form of the MEU model. To be specific, maintaining the CARA form for utility over tokens, the contraction model reduces to:

$$U(\mathbf{x}; \epsilon, \rho) = -\left(\tfrac{1+\epsilon}{3}\right) \exp\{-\rho x_{\min}\} - \tfrac{1}{3} \exp\{-\rho x_2\} - \left(\tfrac{1-\varepsilon}{3}\right) \exp\{-\rho x_{\max}\}.$$

This is exactly the MEU model above with $\theta = \frac{\epsilon}{3}$ and is the kinked specification in equation 1 with $\alpha = \frac{1-\epsilon}{2}$.

[1] The exact formula for integration with respect to a capacity can be found in Schmeidler (1989).

[2] A capacity is convex if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for any sets A and B.

[3] The convex combination of two sets A and B is defined as the union of their pointwise convex combinations: $\lambda A + (1-\lambda)B = \{\lambda a + (1-\lambda)b : a \in A, b \in B\}$.

[1.4] α-Maxmin Expected Utility with fixed priors

A proposed generalization of MEU is α-Maxmin Expected Utility (α-MEU) characterized by Ghirardato et al. (2004) and Olszewski (2006), which evaluates each portfolio by a convex combination of its minimal and maximal expected utilities over some set of subjective prior beliefs over states.

The general form of the α-MEU model is

$$U(\mathbf{x}) = \alpha \cdot \min_{\pi \in \Pi} \int u(x_s)\, d\pi(s) + (1-\alpha) \cdot \max_{\pi \in \Pi} \int u(x_s)\, d\pi(s),$$

where $\Pi \subseteq \Delta S$ is a closed convex set of distributions over states and $\alpha \in [0,1]$ reflects the relative weight of the worst versus the best possible expected utility of $\mathbf{x}$ given Π. Hence, α serves as a parameter reflecting ambiguity aversion. (In the most general case, the α-MEU parameter could depend on the portfolio under consideration $\alpha(\mathbf{x})$.)

If we assume that u has the CARA form and that the set of priors Π is the entire set of distributions consistent with the objective information in the experiment, $\Pi = \{\pi : \pi_2 = \frac{1}{3}\}$, this reduces to the two-parameter formula in equation 1. The weight α and the set of priors Π in the α-MEU model cannot be separately identified. In fact, Siniscalchi (2006) proves that the α-MEU and MEU models are generally confounded in the symmetric case: any MEU representation with some fixed symmetric set of priors can be rewritten as one of a continuum of α-MEU representations with arbitrarily small alternative sets of priors.

In all of the models described above, the parameter α that appears in equation 1 depends on the set Π (or the capacity ν in the case of CEU). Unless the set Π is objectively known, knowledge of the estimated parameter α does not allow us to characterize the degree of ambiguity aversion independently of the degree of ambiguity in the decision problem. In any case, the lack of identification is inherent in these theoretical models, rather than a feature of our data. When we adopt the MEU interpretation, we are fixing $\alpha = 1$ and allowing the set of priors to vary; when we adopt the α-MEU interpretation, we are fixing the set of priors and allowing α to vary. To simplify the exposition and facilitate comparisons, we adopted the second convention as our main interpretation.

[2] The smooth specification

Our second utility specification is differentiable everywhere. The utility of a portfolio $\mathbf{x} = (x_1, x_2, x_3)$ takes the form

$$U(\mathbf{x}; \alpha, \rho) = \frac{1}{\alpha} \int_0^{\frac{2}{3}} -\exp\left\{ -\alpha \left(\begin{array}{c} -\pi_1 \exp\{-\rho x_1\} - \frac{1}{3}\exp\{-\rho x_2\} \\ -\left(\frac{2}{3} - \pi_1\right)\exp\{-\rho x_3\} \end{array} \right) \right\} d\pi_1, \tag{2}$$

This specification involves two iterated integrals. First, the formula inside the parentheses is the expected value of the CARA utility of the portfolio $\mathbf{x}$ when the probability of the first state is known to be π_1. Next, the integral ranging from 0 to $\frac{2}{3}$ takes the expectation of these expected utilities with respect to the uniform distribution for π_1, with each expected utility transformed using a CARA aggregator. The utility function is normalized by $\frac{1}{\alpha}$ so that utility does not go to zero as α approaches zero.

While the kinked specification can be interpreted using a variety of different models, the smooth specification is really motivated by a single model. A recent view of ambiguity aversion (Ergin and

Gul, 2004; Klibanoff et al., 2005; Nau, 2005; and Seo, 2007; as well as related work by Halevy and Feltkamp, 2005; Giraud, 2006; and Ahn, 2008) assumes the agent has a subjective (second-order) distribution μ over the possible (first-order) prior beliefs π over states. Unsure which of the possible first-order prior beliefs actually governs the states, the agent transforms the expected utilities for all prior beliefs π by a concave function φ before integrating these utilities with respect to his second-order distribution μ. This procedure is entirely analogous to the transformation of wealth into cardinal utility before computing expected utility under risk. The concavity of this transformation captures ambiguity aversion. We follow Halevy (2007) in referring to this model as Recursive Expected Utility (REU), owing to its recursive double expectation.

The general form of the REU model is

$$U(\mathbf{x}) = \int_{\Delta S} \varphi \left(\int_S u(x_s)\, d\pi(s) \right) d\mu(\pi),$$

where $\mu \in \Delta(\Delta(S))$ is a (second-order) distribution over possible priors π on S and $\varphi : u(\mathbf{R}_+) \to \mathbf{R}$ is a possibly nonlinear transformation over expected utility levels.[4]

To facilitate comparison with the kinked specification, we reduce the REU model to two parameters. Assuming that

$$\varphi(z) = -e^{-\alpha z},$$

which replicates the constant curvature of u, and that μ is uniformly distributed over the set of priors consistent with the objective information $\Pi = \left\{\pi : \pi_2 = \frac{1}{3}\right\}$, this specializes to the two-parameter formula in equation 2. Here, α reflects the curvature of the aggregator φ and hence measures the degree of ambiguity aversion/seeking: any $\alpha > 0$ indicate preferences that are ambiguity averse, $\alpha < 0$ indicate preferences that are ambiguity seeking, and as $\alpha \to 0$ we approach the standard SEU representation.

One of the crucial features of the REU specification is its reliance on a cardinal utility indicator. Unlike the preferences generated by SEU, MEU and α-MEU, which are invariant to affine transformations of the utility function $u(\cdot)$, the preferences generated by REU are not independent of a change in the scale of utility. For example, if we introduce a scale parameter and set $u(x) = -Ae^{-\rho x}$, the concavity of the transformation φ implies that the ranking of uncertain prospects will not be invariant to changes in A. Since the parameters α and A enter equation 2 only in the form of the product αA, we can estimate αA but cannot identify the values of α and A separately. If we assume a common scale factor for all subjects, say $A = 1$, interpersonal comparisons of ambiguity aversion will still be affected by risk aversion. A higher coefficient of absolute risk aversion, ρ, will reduce the range of the function $u(x) = -e^{-\rho x}$ and, hence, will reduce the ambiguity to which the agent is exposed. We can normalize the ambiguity parameters to take into account the different ranges of expected utility for different subjects, but the meaning of such comparisons is not clear.

[3] Restricted specifications

In addition to the kinked and smooth specifications, we consider two important special cases. The first corresponds to SEU in the sense of Savage, while the second corresponds to an extreme form of MEU. Each is derived by setting the ambiguity parameter equal to some extreme value.

[4] Here, $\Delta(\Delta(S))$ denotes the space of all probability measures over $\Delta(S)$, the set of all probability distributions on S.

[3.1] Ambiguity neutrality: Subjective Expected Utility with a fixed prior

Subjective expected utility (SEU) is a special case of both the kinked and smooth formulations:

$$U(\mathbf{x};\rho) = -\frac{1}{3}\exp\{-\rho x_1\} - \frac{1}{3}\exp\{-\rho x_2\} - \frac{1}{3}\exp\{-\rho x_3\}.$$

This corresponds to the kinked specification in equation 1 with $\alpha = \frac{1}{2}$ and to the smooth specification in equation 2 with $\alpha = 0$ and provides a benchmark for probabilistic sophistication within these specifications.

To derive this formula directly, recall that the general SEU model of Savage (1954) consists of a utility function u which is integrated with respect to a single subjective probability distribution π. The general form for the utility of a portfolio $\mathbf{x} = (x_1, x_2, x_3)$ is:

$$U(\mathbf{x}) = \int_S u(x_s)\, d\pi(s)$$

where π is a subjective probability over states of the world and u is a cardinal utility index over tokens. If we assume that the agent believes the ambiguous states in our experimental choice problem are equally probable, that is, her prior belief over states is $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and has CARA utility over tokens, this specializes to the above formula.

[3.2] Extreme ambiguity aversion: Maxmin Expected Utility with maximal priors

The opposite special case for both the kinked and smooth specifications is the following restricted formulation:

$$U(\mathbf{x};\rho) = -\tfrac{2}{3}\exp\{-\rho x_{\min}\} - \tfrac{1}{3}\exp\{-\rho x_2\}.$$

This corresponds to the kinked specification in equation 1 with $\alpha = 1$ and to the smooth specification in equation 2 as $\alpha \to \infty$ and provides the opposite benchmark of the most ambiguity averse subspecification within these models.

[4] Additional references

1. Hayashi T. and R. Wada (2010) "Choice with Imprecise Information: An Experimental Approach," *Theory and Decision*, *69*, pp. 355-373.

2. Siniscalchi, M. (2006) "A behavioral characterization of plausible priors," *Journal of Economic Theory*, 128, pp. 91-135.