Appendix A.2

NECESSARY AND SUFFICIENT CONDITIONS FOR THE CLASSICAL PROGRAMMING PROBLEM

DANIEL McFADDEN

University of California, Berkeley

We consider the following classical programming problem:

$$\begin{aligned}&\text{Min}\, F(x_1,\ldots,x_n),\\ &\text{s.t. } g^i(x_1,\ldots,x_n)+b_i=0 \quad \text{for} \quad i=1,\ldots,m<n.\end{aligned} \tag{CPP}$$

Letting

$$\underset{1\times n}{\mathbf{x}'} = (x_1,\ldots,x_n),$$

and defining

$$\underset{1\times m}{\mathbf{G}(\mathbf{x})} = (g^1(x),\ldots,g^m(x)),$$

this problem is written in matrix notation as

$$\text{Min}\, F(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{G}(\mathbf{x})+\mathbf{b}'=0.$$

A point $\bar{\mathbf{x}}$ is a *local solution* of CPP if $\mathbf{G}(\bar{\mathbf{x}})+\mathbf{b}'=\mathbf{0}$ and there exists a neighborhood $\mathbf{N}$ of $\bar{\mathbf{x}}$ such that $\mathbf{x}\in\mathbf{N}$ and $\mathbf{G}(\mathbf{x})+\mathbf{b}'=\mathbf{0}$ implies $F(\mathbf{x})\geqq F(\bar{\mathbf{x}})$.

We associate with this problem the Lagrangian

$$L(\mathbf{x},\mathbf{p}) = F(\mathbf{x}) + [\underset{1\times m}{\mathbf{G}(\mathbf{x})}+\mathbf{b}']\underset{m\times 1}{\mathbf{p}},$$

where $\mathbf{p}'=(p_1,\ldots,p_m)$ is a vector of Lagrangian multipliers. We assume hereafter that the functions F, $g^1,\ldots,g^m$ are twice continuously differentiable, and define

$$\underset{n\times 1}{F_{\mathbf{x}}} = \begin{bmatrix} \partial F/\partial x_1 \\ \vdots \\ \partial F/\partial x_n \end{bmatrix},$$

$$\underset{n\times m}{\mathbf{G}_{\mathbf{x}}} = \begin{bmatrix} \partial g^1/\partial x_1 \cdots \partial g^m/\partial x_1 \\ \vdots \qquad\qquad \vdots \\ \partial g^1/\partial x_n \cdots \partial g^m/\partial x_n \end{bmatrix},$$

and

$$F_{\mathbf{xx}} = \begin{bmatrix} \partial^2 F/\partial x_1^2 & \cdots \partial^2 F/\partial x_1 \partial x_n \\ \vdots & \vdots \\ \partial^2 F/\partial x_n \partial x_1 & \cdots \partial^2 F/\partial x_n^2 \end{bmatrix}.$$

Note that

$$L_{\mathbf{x}} = F_{\mathbf{x}} + G_{\mathbf{x}}\mathbf{p} \quad \text{and} \quad L_{\mathbf{p}} = \mathbf{G}(\mathbf{x}) + \mathbf{b}',$$

and that

$$L_{\mathbf{xx}} = F_{\mathbf{xx}} + \sum_{i=1}^{m} g^i_{\mathbf{xx}} p_i.$$

A *Lagrangian critical point* is a vector

$$(\underset{n\times 1}{\bar{\mathbf{x}}}, \underset{m\times 1}{\bar{\mathbf{p}}}),$$

such that

$$L_{\mathbf{x}}(\bar{\mathbf{x}},\bar{\mathbf{p}}) = 0 \quad \text{and} \quad L_{\mathbf{p}}(\bar{\mathbf{x}},\bar{\mathbf{p}}) = 0. \tag{LCP}$$

The first result of classical optimization theory establishes that under mild non-degeneracy conditions on the constraints, each local solution of CPP will correspond to a Lagrangian critical point. We say CPP is *strongly non-degenerate* at a point $\mathbf{x}$ if

$$\text{rank } \mathbf{G}_{\mathbf{x}}(\mathbf{x}) = m. \tag{SND}$$

We say CPP is *weakly non-degenerate* at a point $\mathbf{x}$ if

$$\text{rank } \underset{n\times m}{\mathbf{G}_{\mathbf{x}}(\mathbf{x})} = \text{rank}[\underset{n\times m}{\mathbf{G}_{\mathbf{x}}(\mathbf{x})} \vdots \underset{n\times 1}{F_{\mathbf{x}}(\mathbf{x})}]. \tag{WND}$$

Condition (SND) will hold in most practical programming problems, and can be made to hold in any CPP by an arbitrarily small perturbation of the constraints. Hence, we concentrate our attention on problems satisfying this condition.

Theorem 1. Suppose $\bar{\mathbf{x}}$ is a local solution of CPP, and SND holds at $\bar{\mathbf{x}}$. Then there exists $\bar{\mathbf{p}}$ such that $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP.

The proof of this theorem can be found in many textbooks, and will not be repeated here. See, for example, Intriligator (1971), pp. 31–33).

The next result establishes that at a local solution $\bar{\mathbf{x}}$ of CPP, SND implies WND, and WND holds if and only if $\bar{\mathbf{x}}$ corresponds to a LCP.

Theorem 2. Suppose $\bar{\mathbf{x}}$ is a local solution of CPP. If SND holds at $\bar{\mathbf{x}}$, then WND holds at $\bar{\mathbf{x}}$. There exists $\bar{\mathbf{p}}$ such that $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP if and only if WND holds at $\bar{\mathbf{x}}$.

Proof: If SND holds at $\bar{\mathbf{x}}$, then Theorem 1 implies the existence of $\bar{\mathbf{p}}$ such that $F_{\mathbf{x}}(\bar{\mathbf{x}}) = -\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})\bar{\mathbf{p}}$. But this implies rank $\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}}) = \text{rank}[\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})\vdots F_{\mathbf{x}}(\bar{\mathbf{x}})]$, and WND holds.

The theorem of the alternative for the solution of linear equations states that there exists $\bar{\mathbf{p}}$ such that $F_{\mathbf{x}}(\bar{\mathbf{x}}) = -\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})\bar{\mathbf{p}}$ if and only if condition WND holds. But this is precisely the condition needed for the existence of $\bar{\mathbf{p}}$ such that $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP, since $L_{\mathbf{p}}(\bar{\mathbf{x}},\bar{\mathbf{p}}) = \mathbf{G}(\bar{\mathbf{x}}) + \mathbf{b}' = \mathbf{0}$ is satisfied by assumption. Q.E.D.

The next result establishes that at a strongly non-degenerate local solution of CPP, the Hessian matrix of the Lagrangian, $L_{\mathbf{xx}}$, is positive semidefinite subject to constraint.

Theorem 3. Suppose $\bar{\mathbf{x}}$ is a local solution of CPP and SND holds at $\bar{\mathbf{x}}$. Then, $\mathbf{z}'\mathbf{z} = 1$ and $\mathbf{z}'\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}}) = 0$ imply $\mathbf{z}'L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})\mathbf{z} \geqq 0$, where $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is the LCP whose existence is established by Theorem 1.

Proof: As in the proof of Theorem 1, SND implies that the system of equations

$$\mathbf{G}(\mathbf{v},\mathbf{w}) + \mathbf{b}' = \mathbf{0},$$

where

$$\mathbf{x}' = (\underset{m\times 1}{\mathbf{v}'}, \underset{(n-m)\times 1}{\mathbf{w}'})$$

is a partition of $\mathbf{x}$ such that $\mathbf{G}_{\mathbf{v}}(\bar{\mathbf{v}},\bar{\mathbf{w}})$ is non-singular, has a solution

$$\mathbf{v} = \underset{m\times 1}{\mathbf{h}(\mathbf{w})}$$

in a neighborhood of $\bar{\mathbf{w}}$ satisfying $\bar{\mathbf{v}} = \mathbf{h}(\bar{\mathbf{w}})$ and $\mathbf{G}(\mathbf{h}(\mathbf{w}),\mathbf{w}) + \mathbf{b}' \equiv \mathbf{0}$. Define $f(\mathbf{w}) \equiv F(\mathbf{h}(\mathbf{w}),\mathbf{w})$. Then $\bar{\mathbf{w}}$ is an unconstrained local minimum of $f(\mathbf{w})$. Let

$$\underset{(n-m)\times 1}{\mathbf{y}}$$

be a vector satisfying $\mathbf{y}'\mathbf{y} = 1$, and consider $f(\bar{\mathbf{w}} + \theta\mathbf{y})$ as a function of a scalar θ. A Taylor's expansion in θ yields

$$f(\bar{\mathbf{w}} + \theta\mathbf{y}) - f(\bar{\mathbf{w}}) = \theta\mathbf{y}'f_{\mathbf{w}}(\bar{\mathbf{w}}) + \frac{\theta^2}{2}\mathbf{y}'f_{\mathbf{ww}}(\bar{\mathbf{w}} + \hat{\theta}\mathbf{y})\mathbf{y},$$

where $\hat{\theta}$ is in the interval between 0 and θ. Since $0 \leqq f(\bar{\mathbf{w}} + \theta\mathbf{y}) - f(\bar{\mathbf{w}})$ for θ sufficiently small, we obtain the necessary conditions $f_{\mathbf{w}}(\bar{\mathbf{w}}) = \mathbf{0}$ and $\mathbf{y}'f_{\mathbf{ww}}(\bar{\mathbf{w}})\mathbf{y} \geqq 0$.

Differentiating the identity $g^i(\mathbf{h}(\mathbf{w}),\mathbf{w}) + b_i \equiv 0$, we obtain

$$\underset{(n-m)\times m}{\mathbf{h}'_{\mathbf{w}}}\ \underset{m\times 1}{g^i_{\mathbf{v}}} + \underset{(n-m)\times 1}{g^i_{\mathbf{w}}} \equiv \sum_{j=1}^{m} \underset{(n-m)\times 1}{h^j_{\mathbf{w}}}\ \underset{1\times 1}{g^i_{v_j}} + \underset{(n-m)\times 1}{g^i_{\mathbf{w}}} \equiv \mathbf{0},$$

and

$$0 \equiv \sum_{j=1}^{m} h^j_{\mathbf{ww}} g^i_{v_j} + \sum_{k=1}^{m}\sum_{j=1}^{m} h^j_{\mathbf{w}}(h^k_{\mathbf{w}})' g^i_{v_j v_k} + \sum_{j=1}^{m} [h^j_{\mathbf{w}} g^i_{v_j \mathbf{w}} + g^i_{\mathbf{w}v_j}(h^j_{\mathbf{w}})'] + g^i_{\mathbf{ww}}. \tag{i}$$

Differentiating $f(\mathbf{w}) \equiv F(\mathbf{h}(\mathbf{w}),\mathbf{w})$, we obtain

$$f_{\mathbf{w}} \equiv \mathbf{h}'_{\mathbf{w}}F_{\mathbf{v}} + F_{\mathbf{w}} \equiv \sum_{j=1}^{m} h^j_{\mathbf{w}}F_{v_j} + F_{\mathbf{w}},$$

and

$$f_{\mathbf{ww}} = \sum_{j=1}^{m} h^j_{\mathbf{ww}}F_{v_j} + \sum_{k=1}^{m}\sum_{j=1}^{m} h^j_{\mathbf{w}}(h^k_{\mathbf{w}})'F_{v_j v_k} + \sum_{j=1}^{m} [h^j_{\mathbf{w}}F_{v_j\mathbf{w}} + F_{\mathbf{w}v_j}(h^j_{\mathbf{w}})'] + F_{\mathbf{ww}}. \tag{0}$$

Let $\bar{\mathbf{p}}$ be the vector of Lagrangian multipliers given in the proof of Theorem 1, $\bar{\mathbf{p}} = -G_{\mathbf{v}}(\bar{\mathbf{v}},\bar{\mathbf{w}})^{-1}F_{\mathbf{v}}(\bar{\mathbf{v}},\bar{\mathbf{w}})$. Multiply each equation (i) by $\bar{p}_i$ and add it to equation (0) to obtain

$$f_{\mathbf{ww}} = \sum_{j=1}^{m} h^{j}_{\mathbf{ww}}\left[F_{v_j} + \sum_{i=1}^{m} \bar{p}_i g^{i}_{v_j}\right] + [\underset{(n-m)\times m}{\mathbf{h}'_{\mathbf{w}}} \,\vdots\, \underset{(n-m)^2}{\mathbf{I}_{\mathbf{w}}}] \left[\begin{array}{c|c} L_{\mathbf{vv}} & L_{\mathbf{vw}} \\ \hline L_{\mathbf{wv}} & L_{\mathbf{ww}} \end{array}\right]\left[\begin{array}{c} \mathbf{h}_{\mathbf{w}} \\ \hline \mathbf{I}_{\mathbf{w}} \end{array}\right].$$

But

$$L_{v_j}(\bar{\mathbf{v}},\bar{\mathbf{w}}) = F_{v_j} + \sum_{i=1}^{m} \bar{p}_i g^{i}_{v_j} = 0,$$

and we have

$$\mathbf{y}' f_{\mathbf{ww}} \mathbf{y} = \mathbf{y}'[\mathbf{h}'_{\mathbf{w}} \vdots \mathbf{I}_{\mathbf{w}}] \left[\begin{array}{c|c} L_{\mathbf{vv}} & L_{\mathbf{vw}} \\ \hline L_{\mathbf{wv}} & L_{\mathbf{ww}} \end{array}\right]\left[\begin{array}{c} \mathbf{h}_{\mathbf{w}} \\ \hline \mathbf{I}_{\mathbf{w}} \end{array}\right] \mathbf{y} \geqq 0.$$

But $\mathbf{z}' = (\mathbf{u}',\mathbf{y}')$ satisfying $\mathbf{z}'\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}}) = 0$, or $\mathbf{u}'\mathbf{G}_{\mathbf{v}}(\bar{\mathbf{v}},\bar{\mathbf{w}}) + \mathbf{y}'\mathbf{G}_{\mathbf{w}}(\bar{\mathbf{v}},\bar{\mathbf{w}}) = \mathbf{0}$, implies $\mathbf{u}' = -\mathbf{y}'\mathbf{G}_{\mathbf{w}}\mathbf{G}_{\mathbf{v}}^{-1} = \mathbf{y}'\mathbf{h}'_{\mathbf{w}}$, and hence

$$(\mathbf{u}',\mathbf{y}') \left[\begin{array}{c|c} L_{\mathbf{vv}} & L_{\mathbf{vw}} \\ \hline L_{\mathbf{wv}} & L_{\mathbf{ww}} \end{array}\right]\left[\begin{array}{c} \mathbf{u} \\ \hline \mathbf{y} \end{array}\right] \equiv \mathbf{y}' f_{\mathbf{ww}} \mathbf{y} \geqq 0.$$

This proves that $L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is positive semidefinite subject to the constraint $\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})$; see Section 2 of Appendix A.1. Q.E.D.

The next result shows that a sufficient condition for a LCP $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ to yield a local solution to CPP is that SND hold at $\bar{\mathbf{x}}$ and $L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})$ be positive definite subject to the constraint $\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})$

Theorem 4. Suppose $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP, and suppose SND holds at $\bar{\mathbf{x}}$. If $\mathbf{z}'\mathbf{z} = 1$ and $\mathbf{z}'G_{\mathbf{x}}(\bar{\mathbf{x}}) = \mathbf{0}$ imply $\mathbf{z}'L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})\mathbf{z} > 0$, then $\bar{\mathbf{x}}$ is a local solution of CPP.

Proof: We give a proof by contradiction. Suppose there exist

$$\underset{n\times 1}{\mathbf{z}_k}$$

and θ_k such that $\mathbf{z}'_k\mathbf{z}_k = 1$, $\theta_k \to 0$, $\mathbf{G}(\bar{\mathbf{x}} + \theta_k\mathbf{z}_k) + \mathbf{b}' = 0$, and $F(\bar{\mathbf{x}} + \theta_k\mathbf{z}_k) \leqq F(\bar{\mathbf{x}})$. Without loss of generality, we can assume $\mathbf{z}_k \to \mathbf{z}$. Consider $F(\bar{\mathbf{x}} + \theta\mathbf{z}_k)$ and $g^i(\bar{\mathbf{x}} + \theta\mathbf{z}_k)$ as functions of θ. Taylor's expansions of these functions around $\theta = 0$ yield

$$F(\bar{\mathbf{x}} + \theta_k\mathbf{z}_k) - F(\bar{\mathbf{x}}) = \theta_k\mathbf{z}'_k F_{\mathbf{x}}(\bar{\mathbf{x}}) + \frac{\theta_k^2}{2}\mathbf{z}'_k F_{\mathbf{xx}}(\bar{\mathbf{x}} + \theta_k^0\mathbf{z}_k)\mathbf{z}_k, \tag{0}$$

$$g^i(\bar{\mathbf{x}} + \theta_k\mathbf{z}_k) + b_i = \theta_k\mathbf{z}'_k g^i_{\mathbf{x}}(\bar{\mathbf{x}}) + \frac{\theta_k^2}{2}\mathbf{z}'_k g^i_{\mathbf{xx}}(\bar{\mathbf{x}} + \theta_k^i\mathbf{z}_k)\mathbf{z}_k, \tag{i}$$

where θ_k^0, θ_k^i are in the interval between 0 and θ_k, and we have used $g^i(\bar{\mathbf{x}})+b_i=0=L_{p_i}(\bar{\mathbf{x}},\bar{\mathbf{p}})$. Multiplying equation (i) above by $\bar{p}_i$ for each i and adding it to equation (0) yields, since $g^i(\bar{\mathbf{x}}+\theta_k\mathbf{z}_k)+b_i=0$,

$$0 \geqq F(\bar{\mathbf{x}}+\theta_k\mathbf{z}_k)-F(\bar{\mathbf{x}})$$
$$=\theta_k\mathbf{z}_k'L_{\mathbf{x}}(\bar{\mathbf{x}},\bar{\mathbf{p}})+\frac{\theta_k^2}{2}\mathbf{z}_k'\left[F_{\mathbf{xx}}(\bar{\mathbf{x}}+\theta_k^0\mathbf{z}_k)+\sum_{i=1}^m g_{\mathbf{xx}}^i(\bar{\mathbf{x}}+\theta_k^i\mathbf{z}_k)\bar{p}_i\right]\mathbf{z}_k,$$

or dividing by θ_k^2 and taking the limit $\mathbf{z}_k\to\mathbf{z}$, $\theta_k\to 0$,

$$0\geqq \mathbf{z}'L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})\mathbf{z}.$$

But dividing (i) by θ_k and taking the limit $\mathbf{z}_k\to\mathbf{z}$, $\theta_k\to 0$ yields, since $g^i(\bar{\mathbf{x}}+\theta_k\mathbf{z}_k)+b_i=0$, $\mathbf{z}'g_x^i(\bar{\mathbf{x}})=\mathbf{0}$. Then $\mathbf{z}'\mathbf{z}=1$, $\mathbf{z}'\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})=0$, and $\mathbf{z}'L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})\mathbf{z}\leqq 0$ contradicts the hypothesis. Q.E.D.

Lemma 4 in Appendix A.1 provides the following reformulation:

Theorem 5. Suppose $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP, and suppose in the matrix

$$\left[\begin{array}{c|c}\underset{n\times m}{L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})} & \underset{n\times m}{\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})}\\ \hline \underset{m\times n}{\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})'} & \underset{m\times m}{\mathbf{0}}\end{array}\right],$$

there exists a nested sequence of principal minors with the sign $(-1)^m$ formed by deleting r symmetric rows and columns from the first n, for $r=0,\ldots,n-m$. Then SND holds and $\bar{\mathbf{x}}$ is a local solution of CPP.

Proof: For $r=n-m$, the principal minor above is, except for sign, the squared determinant of a $m\times m$ submatrix of $\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})$, establishing that SND holds. Lemma 4 in Appendix A.1 establishes the result. Q.E.D.

A series of examples demonstrate the role of the assumptions in Theorems 1–4.

Example 1. Min $x_1+x_2^2$ subject to $x_1^2=0$. The minimum is at $\bar{x}_1=\bar{x}_2=0$ where $F_{\mathbf{x}}(\bar{\mathbf{x}})'=(1,0)$ and $g_{\mathbf{x}}^1(\bar{\mathbf{x}})'=(0,0)$. Then rank $g_{\mathbf{x}}^1(\bar{\mathbf{x}})<$ rank $(g_{\mathbf{x}}^1(\bar{\mathbf{x}})\ F_{\mathbf{x}}(\bar{\mathbf{x}}))$ and WND fails, so the Lagrangian method cannot be applied.

Example 2. Min $-x_1^2+x_2^2$ subject to $x_1^2=0$. The minimum is at $\bar{x}_1=\bar{x}_2=0$ where

$$F_{\mathbf{x}}(\bar{\mathbf{x}})' = (0,0) = g^1_{\mathbf{x}}(\bar{\mathbf{x}})',$$

and WND holds. Then $(\bar{\mathbf{x}},\bar{p})$ is a LCP for any $\bar{p}$, and

$$L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}}) = \begin{bmatrix} 2(\bar{p}-1) & 0 \\ 0 & 2 \end{bmatrix}.$$

For $\bar{p} < 1$, $L_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{p})$ is not positive semidefinite subject to $g^1_{\mathbf{x}}(\bar{\mathbf{x}})$. Hence, WND cannot replace SND in Theorem 3.

Example 3. Min $-x_1^2 + x_2^2$ subject to $-x_1^2 + x_2^2 = 0$. A minimum is at $\bar{x}_1 = \bar{x}_2 = 1$, where $g^1_{\mathbf{x}}(\bar{\mathbf{x}})' = (-2,2)$ and SND is satisfied. Then $(\bar{\mathbf{x}},\bar{p})$ with $\bar{p} = -1$ is a LCP and $L_{\mathbf{xx}}(\bar{x},\bar{p})$ is the zero matrix. Then $L_{\mathbf{xx}}$ is positive semidefinite, but not positive definite, subject to constraint.

Example 4. Min $x_1^2 + 2x_1x_2 + x_2^2 + x_3^2$ subject to $x_1 - x_2 = 0$. The minimum occurs at $\bar{\mathbf{x}} = \mathbf{0}$, where

$$g_{\mathbf{x}}(\bar{\mathbf{x}})' = (1,-1,0),$$

$$F_{\mathbf{x}}(\bar{\mathbf{x}})' = (0,0,0),$$

and

$$F_{\mathbf{xx}}(\bar{\mathbf{x}}) = L_{\mathbf{xx}}(\bar{\mathbf{x}}) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then SND holds and $\bar{p} = 0$. $L_{\mathbf{xx}}(\bar{\mathbf{x}})$ is positive semidefinite, and subject to the constraint $g_{\mathbf{x}}(\bar{\mathbf{x}})'\mathbf{z} = 0 = z_1 = z_2$ is positive definite.

A final comment regards the maximization problem

$$\text{Max}\, F(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{G}(\mathbf{x}) + \mathbf{b}' = \mathbf{0}. \tag{CPP2}$$

This is equivalent to the minimization problem

$$\text{Min}\, [-F(\mathbf{x})] \quad \text{s.t.} \quad -\mathbf{G}(\mathbf{x}) - \mathbf{b}' = \mathbf{0}, \tag{*}$$

to which Theorems 1 and 5 apply. Hence, defining $L^*(\mathbf{x},\mathbf{p}) = F(\mathbf{x}) + [\mathbf{G}(\mathbf{x}) + \mathbf{b}']\mathbf{p}$, we have the following result:

Theorem 6. Suppose $\bar{\mathbf{x}}$ is a local solution of CPP2 and SND holds at $\bar{\mathbf{x}}$. Then there exists $\bar{\mathbf{p}}$ such that $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP. Alternately, suppose $(\bar{\mathbf{x}},\bar{\mathbf{p}})$ is a LCP, and suppose in the matrix

$$\left[\begin{array}{c|c} \underset{n\times n}{L^*_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}})} & \underset{n\times m}{\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})} \\ \hline \underset{m\times n}{\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})'} & \underset{m\times m}{\mathbf{0}} \end{array}\right],$$

there exists a nested sequence of principal minors with the sign $(-1)^{n-r}$ formed by deleting r symmetric rows and columns from the first n, for $r = 0,\ldots,n - m$. Then SND holds and $\bar{\mathbf{x}}$ is a local solution of CPP2.

Proof: The proof of Theorem 1 applies without modification to the first part of this theorem. To prove the second part, we apply Theorem 5 to the minimization problem (*), obtaining the sufficient condition for $\bar{\mathbf{x}}$ to be a local solution that

$$\left|\begin{array}{c|c} -L^{*}_{\mathbf{xx}}(\bar{\mathbf{x}},\bar{\mathbf{p}}) & -\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}}) \\ \hline -\mathbf{G}_{\mathbf{x}}(\bar{\mathbf{x}})' & \mathbf{0} \end{array}\right|,$$

have a nested sequence of principal minors of sign $(-1)^m$. Reversing the sign of all rows in the principal minor formed by deleting r rows and columns multiplies the determinant by the factor $(-1)^{n+m-r}$; it is then required to have the sign $(-1)^{n-r+2m} = (-1)^{n-r}$. But these are just the principal minors considered in the statement of the theorem, giving the desired result. Q.E.D.