Problem Set #3
Suggested Solutions

1 **(A)** The given problem is:

$$\max_{(Y,H)} \ u(Y,H) = \ln Y + \frac{H}{12}$$

s.t.

$$Y = \left(6(24-H)\right)^{\frac{1}{2}} \qquad \text{(I)}$$

Using the equality constraint we can eliminate Y from the objective.

$$\max_{H} \ u(H) = \ln\left(6(24-H)\right)^{\frac{1}{2}} + \frac{H}{12}$$

Since

$$\frac{\partial u}{\partial H} = -\frac{6\left(6(24-H)\right)^{-\frac{1}{2}}}{2\left(6(24-H)\right)^{\frac{1}{2}}} + \frac{1}{12} = -\left[\frac{3}{6(24-H)} - \frac{1}{12}\right] = -\left[\frac{1}{2(24-H)} - \frac{1}{12}\right]$$

The optimal choice of H will be given by the critical point:

$$\frac{\partial u}{\partial H} = 0 \Leftrightarrow \frac{1}{24-H} - \frac{1}{6} = 0 \Leftrightarrow H^* = 18$$

This is Mr. Crusoe's chosen amount of hours of leisure (i.e. he would choose to work 6 hours).

From the equality constraint (I) we get the amount of yams that Mr. Crusoe can consume when he chooses to leisure for H^* hours.

$$Y^* = \left(6(24-H^*)\right)^{\frac{1}{2}} = 6$$

At this optimal choice $\left(Y^*, H^*\right)$, the maximized level of utility that Mr. Crusoe enjoys is:

$$u\left(Y^*, H^*\right) = \ln Y^* + \frac{H^*}{12} = \ln 6 + \frac{18}{12} = \ln 6 + \frac{3}{2}$$

(B) The problem now becomes:

$$\max_{(Y,H)} \ u(Y,H) = \ln Y + \frac{H}{12}$$

s.t.

$$Y = \left(6(24-H)\right)^{\frac{1}{2}}$$

$$Y \geq 0$$

$$H \geq 0$$

$$H \leq 24$$

The Lagrangean function can be written as:

$$\tilde{L}(Y, H, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = u(Y, H) + \lambda_1\left((6(24-H))^{\frac{1}{2}} - Y\right) + \lambda_2(Y-0) + \lambda_3(H-0) + \lambda_4(24-H)$$

$$\tilde{L} = \ln Y + \frac{H}{12} + \lambda_1\left((6(24-H))^{\frac{1}{2}} - Y\right) + \lambda_2(Y-0) + \lambda_3(H-0) + \lambda_4(24-H)$$

The first order conditions would be

<u>Set (I)</u>

$$\frac{\partial \tilde{L}}{\partial Y} = 0 \quad \text{or} \quad \frac{1}{Y} - \lambda_1 + \lambda_2 = 0 \tag{1}$$

$$\frac{\partial \tilde{L}}{\partial H} = 0 \quad \text{or} \quad \frac{1}{12} + \frac{3\lambda_1}{(6(24-H))^{\frac{1}{2}}} + \lambda_3 - \lambda_4 = 0 \tag{2}$$

<u>Set (II)</u>

$$\frac{\partial \tilde{L}}{\partial \lambda_1} = 0 \quad \text{or} \quad Y = (6(24-H))^{\frac{1}{2}} \tag{3}$$

$$\frac{\partial \tilde{L}}{\partial \lambda_2} \geq 0 \quad \text{or} \quad Y \geq 0 \tag{4.1}$$

$$\lambda_2 \geq 0 \tag{4.2}$$

$$\lambda_2 \frac{\partial \tilde{L}}{\partial \lambda_2} = 0 \quad \text{or} \quad \lambda_2 Y = 0 \tag{4.3}$$

$$\frac{\partial \tilde{L}}{\partial \lambda_3} \geq 0 \quad \text{or} \quad H \geq 0 \tag{5.1}$$

$$\lambda_3 \geq 0 \tag{5.2}$$

$$\lambda_3 \frac{\partial \tilde{L}}{\partial \lambda_3} = 0 \quad \text{or} \quad \lambda_3 H = 0 \tag{5.3}$$

$$\frac{\partial \tilde{L}}{\partial \lambda_4} \geq 0 \quad \text{or} \quad H \leq 24 \tag{6.1}$$

$$\lambda_4 \geq 0 \tag{6.2}$$

$$\lambda_4 \frac{\partial \tilde{L}}{\partial \lambda_4} = 0 \quad \text{or} \quad \lambda_4 (24-H) = 0 \tag{6.3}$$

These definitely look too many and cumbersome to work out in order to get the Lagrangean Critical Point (LCP) but we can try and be a bit clever about it by thinking about the economic problem at hand rather than getting lost in the math.
By inspection of the utility function to be maximized, it is obvious that the agent would never choose $Y = 0$ since this gives him an infinitely-negative utility level. Since, we must satisfy the non-negativity constraint (4.1) on *Y*, it must be

$Y > 0$ (I)

But then

$(4.3) \overset{(I)}{\Rightarrow} \lambda_2 = 0$

Equation (1) now gives: $\lambda_1 = \frac{1}{Y} > 0$

Moreover, note that, because of (I), we cannot have $H = 24$. Hence,

$H < 24$

and (6.3) gives

$\lambda_4 = 0$

The trick with these Lagrangean problems is to try and establish as many equations as the ones above as possible using simple economic argument to start with like the one we employed here. Once one has eliminated all possibilities of proceeding further along the lines of such arguments, one must come back to face the tedious algebraic manipulations.

We have established that at a LCP, *Y* must be strictly positive. Yet we were not able to determine anything about the other choice variable *H*. Therefore, we need to consider all possibilities for *H* that are embedded within satisfying its own non-negativity constraint (5.1).

<u>Case 1</u>: $H > 0$

If $H > 0$ at the LCP, then (5.3) gives $\lambda_3 = 0$

Equations (1) and (2) simplify to:

$$\lambda_1 = \frac{1}{Y} \qquad (1)$$

$$\frac{1}{12} + \frac{3\lambda_1}{(6(24 - H))^{\frac{1}{2}}} = 0 \qquad (2)$$

Along with equation (3), we now have a *3x3* system of equations to solve in order to determine the LCP.

Solution:

$Y = 6 \;\; H = 18 \;\; \lambda_1 = \frac{1}{6} \;\; \lambda_2 = \lambda_3 = \lambda_4 = 0$

Case 2: $H = 0$
Then (3) gives $Y = 12$

Equation (1): $\lambda_1 = \frac{1}{12}$

Equation (2): $\frac{1}{12} + \frac{\frac{3}{12}}{(6(24-0))^{\frac{1}{2}}} + \lambda_3 = 0 \Leftrightarrow \frac{15}{144} + \lambda_3 = 0$

But this result for the multiplier λ_3 violates its non-negativity condition (5.3). Hence, this case does not lead to a solution for a LCP.

We conclude therefore that there is a unique LCP in this problem given by case 1.

(C) We know that a Lagrangean multiplier is associated with a given constraint - say $g(Y,H) \le c$ in a two-choice-variable problem. Moreover, we know that

$\lambda_1^* = \left.\frac{\partial u(Y,H)}{\partial c}\right|_{(Y^*,H^*)}$ where stars indicate optimal values.

In this example, the economic interpretation of the Lagrangean multiplier is that it is the ***shadow price*** of its corresponding constraint. This means that Robinson would be willing to pay $\lambda_1^* = \frac{1}{12}$ in order to have the production constraint (I) relaxed by one unit. Since, in this problem everythin is measured in units of Yam production, at his optimal point when he is working 6 hours and consuming 6 units of Yams, he is willing to pay 1/12 units of Yams in order to have the constraint relaxed (i.e. be able to move Y in the direction $Y > (6(24-H))^{\frac{1}{2}}$) by one more of Yams.

The LCP is only a ***necessary*** condition for a solution to the UMP. You should verify, however, that, in this problem, the constrained objective function is concave (i.e. evaluate the bordered Hessian at the LCP and show that it is negative definite). This ensures (as a second order sufficient condition) that our LCP is indeed a maximizer.

2 The given problem is:

$\max_{(Y,L)} \pi(Y,L) = Y - wL$

s.t.

$Y = (6L)^{\frac{1}{2}}$ (I)

Using the equality constraint we can eliminate *Y* from the objective.

$\max_{L} \pi(L) = (6L)^{\frac{1}{2}} - wL$

Since

$$\frac{\partial \pi}{\partial L} = \frac{3}{(6L)^{\frac{1}{2}}} - w$$

The optimal choice of L will be given by the critical point:

$$\frac{\partial \pi}{\partial L} = 0 \Leftrightarrow$$

$$\frac{3}{(6L)^{\frac{1}{2}}} - w = 0 \Leftrightarrow$$

$$L^* = \frac{3}{2w^2}$$

This is Mr. Friday's demand for labor as a function of the given wage rate.

From the equality constraint (I) we get the amount of yams supplied when the amount $L(w)$ of labor hours is employed (and, consequently, when the given wage rate is w):

$$Y^* = \left(\frac{18}{2w^2}\right)^{\frac{1}{2}} = \frac{3}{w}$$

At the given wage rate w, the maximum level of profit that Mr. Friday can achieve is:

$$\pi^* = Y^* - wL^* = Y(w) - wL(w) = \frac{3}{w} - w\left(\frac{3}{2w^2}\right) = \frac{3}{2w}$$

3 The profit function that we derived in the previous question is:

$$\pi(w) = \frac{3}{2w}$$

Since $\frac{\partial \pi}{\partial w} = -\frac{3}{2w^2} < 0$, the profit function is non-increasing.

Also $\frac{\partial^2 \pi}{\partial w^2} = \frac{3}{w^3} > 0$. Hence, the profit function is convex.

It is obvious that: $\frac{\partial \pi}{\partial w} = -L(w)$

4 Robinson's problem now becomes:

$$\max_{(Y,H)} \; u(Y,H) = \ln Y + \frac{H}{12}$$

s.t.

$$Y = \pi + w(24 - H)$$
$$Y \geq 0$$
$$H \geq 0$$
$$H \leq 24$$

The Lagrangean function can be written as:

$$\tilde{L}(Y,H,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = u(Y,H) + \lambda_1(\pi + w(24-H) - Y) + \lambda_2(Y-0) + \lambda_3(H-0) + \lambda_4(24-H)$$

$$\tilde{L} = \ln Y + \frac{H}{12} + \lambda_1(\pi + w(24-H) - Y) + \lambda_2(Y-0) + \lambda_3(H-0) + \lambda_4(24-H)$$

The first order conditions would be

<u>Set (I)</u>

$$\frac{\partial \tilde{L}}{\partial Y} = 0 \quad \text{or} \quad \frac{1}{Y} - \lambda_1 + \lambda_2 = 0 \qquad (1)$$

$$\frac{\partial \tilde{L}}{\partial H} = 0 \quad \text{or} \quad \frac{1}{12} - w\lambda_1 + \lambda_3 - \lambda_4 = 0 \qquad (2)$$

<u>Set (II)</u>

$$\frac{\partial \tilde{L}}{\partial \lambda_1} = 0 \quad \text{or} \quad Y = \pi + w(24 - H) \qquad (3)$$

$$\frac{\partial \tilde{L}}{\partial \lambda_2} \geq 0 \quad \text{or} \quad Y \geq 0 \qquad (4.1)$$

$$\lambda_2 \geq 0 \qquad (4.2)$$

$$\lambda_2 \frac{\partial \tilde{L}}{\partial \lambda_2} = 0 \quad \text{or} \quad \lambda_2 Y = 0 \qquad (4.3)$$

$$\frac{\partial \tilde{L}}{\partial \lambda_3} \geq 0 \quad \text{or} \quad H \geq 0 \qquad (5.1)$$

$$\lambda_3 \geq 0 \qquad (5.2)$$

$\lambda_3 \dfrac{\partial \tilde{L}}{\partial \lambda_3} = 0$ or $\lambda_3 H = 0$ (5.3)

$\dfrac{\partial \tilde{L}}{\partial \lambda_4} \geq 0$ or $H \leq 24$ (6.1)

$\lambda_4 \geq 0$ (6.2)

$\lambda_4 \dfrac{\partial \tilde{L}}{\partial \lambda_4} = 0$ or $\lambda_4 (24 - H) = 0$ (6.3)

Using the same economic argument we did in problem 1, we conclude that:

$Y > 0 \quad \lambda_2 = 0 \quad \lambda_1 = \dfrac{1}{Y} > 0$

However, in this case, we cannot necessarily claim $H < 24$ as we did in problem 1. Here, the agent receives dividends π and it is possible, if these are high enough, that he will optimally chose $H = 24$ (i.e. not to work at all)[1].

Consider the following possibilities:

Case 1: $0 < H < 24$

If $H > 0$ at the LCP, then (5.3) gives $\lambda_3 = 0$

Similarly, $H < 24$ gives via (6.3) $\lambda_4 = 0$

Equations (1) and (2) simplify to:

$\lambda_1 = \dfrac{1}{Y}$ (1)

$\dfrac{1}{12} - w\lambda_1 = 0$ (2)

Along with equation (3), we now have a *3x3* system of equations to solve in order to determine the LCP.

Candidate Solution for UMP:

$Y = 12w \quad H = 12 + \dfrac{\pi}{w} \quad \lambda_1 = \dfrac{1}{12w} \quad \lambda_2 = \lambda_3 = \lambda_4 = 0$

In this solution,

- Yam demand: $Y(\pi, w) = 12w$
- Labor supply: $L(\pi, w) = 24 - H = 12 - \dfrac{\pi}{w}$

[1] Recall our reasoning for the claim in problem 1 - $H = 24 \Rightarrow Y = 0$. Here, $H = 24 \Rightarrow Y = \pi > 0$ which does not violate the non-zero condition on Y imposed by the given utility function.

- Maximized utility level:
  $$u\left(Y^*, H^*\right) = u\left(Y(\pi, w), H(\pi, w)\right) = \ln(12w) + \frac{1}{12}\left(12 + \frac{\pi}{w}\right) = 1 + \ln(12w) + \frac{\pi}{12w}$$
  Note that the latter function is called the ***indirect utility function*** and is usually denoted by $V(\pi, w)$

<u>Case 2</u>: $H = 0$
Since $H < 24$, (6.3) gives $\lambda_4 = 0$

Then (3) gives $Y = \pi + 24w$
Equation (1): $\lambda_1 = \dfrac{1}{\pi + 24w}$
Equation (2): $\dfrac{1}{12} - \dfrac{w}{\pi + 24w} + \lambda_3 = 0 \Leftrightarrow \lambda_3 = -\dfrac{\pi + 12w}{12(\pi + 12w)} < 0$
This result for the multiplier λ_3 violates its non-negativity condition (5.2).
Hence, this case does not lead to a solution for a LCP.

<u>Case 3</u>: $H = 24$
Since $H > 0$, (5.3) gives $\lambda_3 = 0$

(3) gives $Y = \pi$
Equation (1): $\lambda_1 = \dfrac{1}{\pi}$
Equation (2): $\dfrac{1}{12} - w\lambda_1 - \lambda_4 = 0 \Leftrightarrow \lambda_4 = \dfrac{1}{12} - \dfrac{w}{\pi}$
This result for the multiplier λ_4 is in agreement with its non-negativity condition (6.2) if and only if $\pi \geq 12w$.
Finally, it remains for us to determine the value of the multiplier λ_2. Since $Y = \pi > 0$, the complementary slackness condition (4.3) gives $\lambda_2 = 0$.

<u>*Candidate Solution for UMP:*</u>
$Y = \pi \quad H = 24 \quad \lambda_1 = \dfrac{1}{\pi} \quad \lambda_2 = \lambda_3 = 0 \quad \lambda_4 = \dfrac{1}{12} - \dfrac{w}{\pi}$ valid iff $\pi \geq 12w$

In this solution,

- Yam demand: $Y(\pi, w) = \pi$
- Labor supply: $L(\pi, w) = 0$
- Maximized utility level: $V(\pi, w) = u\left(Y(\pi, w), L(\pi, w)\right) = 2 + \ln \pi$

In order to determine which of our two candidate solutions (the two LCPs) is the actual maximizer, we need to compare their respective derived utility levels.

We have

$$1+\ln(12w)+\frac{\pi}{12w}\geq 2+\ln\pi \Leftrightarrow$$

$$\frac{\pi}{12w}-1\geq \ln\left(\frac{\pi}{12w}\right)\Leftrightarrow$$

$$e^{\frac{\pi}{12w}-1}\geq \frac{\pi}{12w}\Leftrightarrow \qquad (*)^2$$

$$\frac{1+\frac{\pi}{12w}}{e}\geq \frac{\pi}{12w}\Leftrightarrow$$

$$\pi\leq \frac{12w}{e-1}$$

But recall that the LCP of case 3 is valid only when $\pi \geq 12w$. Hence, we conclude:

<u>*Solution of the UMP:*</u>

- Yam demand: $Y(\pi,w)=\begin{cases}12w & \pi<12w\\ \pi & \pi\geq 12w\end{cases}$
- Labor supply: $L(\pi,w)=\begin{cases}12-\dfrac{\pi}{w} & \pi<12w\\ 0 & \pi\geq 12w\end{cases}$
- Maximized utility level[3]: $V(\pi,w)=\begin{cases}1+\ln(12w)+\dfrac{\pi}{12w} & \pi<12w\\ 2+\ln\pi & \pi\geq 12w\end{cases}$

Roy's Identity

$$\frac{\frac{\partial V}{\partial w}}{\frac{\partial V}{\partial \pi}}=\begin{cases}\dfrac{\frac{1}{w}-\frac{\pi}{12w^2}}{\frac{1}{12w}} & \pi<12w\\ 0 & \pi\geq 12w\end{cases}=\begin{cases}12-\dfrac{\pi}{w} & \pi<12w\\ 0 & \pi\geq 12w\end{cases}=L(\pi,w)$$

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[2] I am using the Taylor expansion $f(x)=e^x\approx 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+..$ around the point *x=0*

[3] Note that at $\pi=12w$ the upper and lower equations coincide. Hence, $V(\pi,w)$ is continuous in the entire domain for *w*.

5 From problem 2, the demand for labor function was found to be:

$$L_D(w) = \frac{3}{2w^2}$$

The labor supply, on the other hand is:

$$L_S(\pi, w) = \begin{cases} 12 - \dfrac{\pi}{w} & \pi < 12w \\ 0 & \pi \geq 12w \end{cases}$$

For the labor market to clear, we obviously must be in the $\pi < 12w$ range. Hence:

$$L_D(w) = L_S(w) \Leftrightarrow$$

$$\frac{3}{2w^2} = 12 - \frac{\pi}{w} \Leftrightarrow$$

$$\frac{3}{2w^2} = 12 - \frac{3}{2w^2} \Leftrightarrow$$

$$w^* = \frac{1}{2}$$

At this equilibrium wage rate, the profits that Mr. Friday will be giving to Mr. Crusoe will be[4]: $\pi^* = \frac{3}{2w^*} = 3$

Note that $\pi^* < 12w^*$ and Mr. Crusoe will, therefore, choose to work.

He will offer to work $L_S^* = 12 - \frac{\pi^*}{w^*} = 6$ hours and, subsequently, leisure for $H^* = 24 - L^* = 18$ hours. His demand for Yams will be $Y^* = Y(\pi^*, w^*) = 12w^* = 6$.

These are exactly the same choices for (Y, H) that Robinson made in problem 1. Clearly, he will achieve the same level of utility.

[4] You should verify that Walras' Law holds. In this economy we have only two markets – the labor and output (Yams) markets. We have shown in the text that the former market clears at the wage rate *w**. Check that this particular *w** manages to also equate the quantity of Yams that Mr. Friday wishes to supply (problem 2) with the quantity that Robinson demands (problem 4).