Problem Set 4
Suggested Solutions

Problem 1

(A) The market demand function is the solution to the following utility-maximization problem (UMP):

$$\max_{(x_1,x_2,x_3)} U(x_1,x_2,x_3) = (x_1x_2)^{\frac{1}{3}} + x_3$$
$$s.t.$$
$$p_1x_1 + p_2x_2 + p_3x_3 \le y$$
$$x_1, x_2, x_3 \ge 0$$

The Lagrangean:

$$L(x_1,x_2,x_3;\lambda,\mu_1,\mu_2,\mu_3) = (x_1x_2)^{\frac{1}{3}} + x_3 + \lambda[y - p_1x_1 - p_2x_2 - p_3x_3]$$
$$+\mu_1(x_1-0) + \mu_2(x_2-0) + \mu_3(x_3-0)$$

The first-order conditions:

Set (I)

$$\frac{\partial L(\underline{x};\lambda,\mu_1,\mu_2,\mu_3)}{\partial x_1} = 0 \Leftrightarrow \frac{\partial U(\underline{x})}{\partial x_1} = \lambda p_1 - \mu_1 \Leftrightarrow \lambda p_1 - \mu_1 = \frac{1}{3}(x_1x_2)^{-\frac{2}{3}} x_2 \qquad (1.1)$$

$$\frac{\partial L(\underline{x};\lambda,\mu_1,\mu_2,\mu_3)}{\partial x_2} = 0 \Leftrightarrow \frac{\partial U(\underline{x})}{\partial x_2} = \lambda p_2 - \mu_2 \Leftrightarrow \lambda p_2 - \mu_2 = \frac{1}{3}(x_1x_2)^{-\frac{2}{3}} x_1 \qquad (1.2)$$

$$\frac{\partial L(\underline{x};\lambda,\mu_1,\mu_2,\mu_3)}{\partial x_3} = 0 \Leftrightarrow \frac{\partial U(\underline{x})}{\partial x_3} = \lambda p_3 - \mu_3 \Leftrightarrow \lambda p_3 - \mu_3 = 1 \qquad (1.3)$$

Set (II)

$$p_1x_1 + p_2x_2 + p_3x_3 \le y \qquad (2.1)$$
$$\lambda \ge 0 \qquad (2.2)$$
$$\lambda\left[y - p_1x_1 - p_2x_2 - p_3x_3\right] = 0 \qquad (2.3)$$

$$x_1 \ge 0 \qquad \mu_1 \ge 0 \qquad \mu_1 x_1 = 0 \qquad (3.1)$$

$$x_2 \geq 0 \qquad \mu_2 \geq 0 \qquad \mu_2 x_2 = 0 \qquad (3.2)$$

$$x_3 \geq 0 \qquad \mu_3 \geq 0 \qquad \mu_3 x_3 = 0 \qquad (3.3)$$

We have the following possible cases:

- (Interior Solution) $x_1, x_2, x_3 > 0 \overset{(3.1),(3.2),(3.3)}{\Rightarrow} \mu_1 = \mu_2 = \mu_3 = 0$

By (1.3), since we are given $p_i > 0 \ \ \forall i = 1,2,3$, we get $\lambda = \frac{1}{p_3} > 0$.

Hence, by (2.3): $p_1 x_1 + p_2 x_2 + p_3 x_3 = y$

$$\frac{(1.1)}{(1.2)} \rightarrow \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

From (1.1):

$$\frac{p_1}{p_3} = \frac{1}{3}\left(\frac{x_2}{x_1^2}\right)^{\frac{1}{3}} = \frac{1}{3}\left(\frac{\frac{x_1 p_1}{p_2}}{x_1^2}\right)^{\frac{1}{3}} = \frac{1}{3}\left(\frac{p_1}{x_1 p_2}\right)^{\frac{1}{3}} \Rightarrow$$

$$x_1 = \frac{1}{27}\left(\frac{p_3^3}{p_1^2 p_2}\right)$$

Thus,

$$x_2 = \frac{1}{27}\left(\frac{p_3^3}{p_1 p_2^2}\right)$$

Finally: $p_1 x_1 + p_2 x_2 + p_3 x_3 = y \Rightarrow x_3 = \frac{y - 2 p_1 x_1}{p_3} = \frac{y}{p_3} - \frac{2 p_3^2}{27 p_1 p_2}$

Hence, a candidate solution is:

$$x^1(p_1, p_2, p_3, y) = \begin{pmatrix} \frac{1}{27}\left(\frac{p_3^3}{p_1^2 p_2}\right) \\ \frac{1}{27}\left(\frac{p_3^3}{p_1 p_2^2}\right) \\ \frac{y}{p_3} - \frac{2}{27}\left(\frac{p_3^2}{p_1 p_2}\right) \end{pmatrix}$$

- $x_1 = x_2 = 0, \ \ x_3 > 0$

Thus, $\mu_3 = 0$ and by (1.3): $\lambda = \frac{1}{p_3} > 0$

Hence, by (2.3): $p_3x_3 = y \Rightarrow x_3 = \frac{y}{p_3}$

From (1.1): $\mu_1 = \frac{p_1}{p_3} > 0$ Similarly, from (1.2): $\mu_2 = \frac{p_2}{p_3} > 0$

Hence, a candidate solution for the UMP is: $x^2(p_1, p_2, p_3, y) = \begin{pmatrix} 0 \\ 0 \\ \frac{y}{p_3} \end{pmatrix}$

- $x_1, x_2 > 0, \quad x_3 = 0$

Thus, $\mu_1 = \mu_2 = 0$ and by (1.3) or (1.2): $\lambda > 0$

Hence, by (2.3): $p_1x_1 + p_2x_2 = y$

$\frac{(1.1)}{(1.2)} \rightarrow \frac{x_2}{x_1} = \frac{p_1}{p_2}$

Hence, $x_1 = \frac{y}{2p_1}$ and $x_2 = \frac{y}{2p_2}$

But from (1.1): $\mu_1 = \frac{p_1}{p_3} > 0$ since the price vector is strictly positive. This result clearly violates our earlier premise that $\mu_1 = 0$.

Therefore, this case cannot yield a candidate solution[1].

Note that we don't need to check the sub-cases of each of the cases (a) $x_1 = x_2 = 0, \quad x_3 > 0$ and (b) $x_1, x_2 > 0, \quad x_3 = 0$ where one of x_1, x_2 is strictly positive while the other being zero because, by the functional form of the given utility function, the zero element drives the entire first term in the utility function to zero irrespective of the fact that the other element is non-zero. Hence, if for example $x_1 = 0$, there is no point is wasting money on x_2. In other words, (i) $x_1 = 0, \quad x_2 > 0$ and, similarly, (ii) $x_2 = 0, \quad x_1 > 0$ could never be optimal.

It is easy to verify that, from the two candidate solutions that we found above, the first one yields the highest level of utility. Specifically,

$$U\left(x^1(p_1, p_2, p_3, y)\right) = \frac{y}{p_3} + \frac{p_3^2}{27p_1p_2} > \frac{y}{p_3} = U\left(x^2(p_1, p_2, p_3, y)\right)$$

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[1] One would reach the same conclusion by noticing instead that from (1.2): $\mu_2 = \frac{p_2}{p_3} > 0$.

Finally, the Marshallian demand is given:

$$x\left(p_1,p_2,p_3,y\right)=\begin{pmatrix}\dfrac{p_3^3}{27p_1^2p_2}\\ \dfrac{p_3^3}{27p_1p_2^2}\\ \dfrac{y}{p_3}-\dfrac{2p_3^2}{27p_1p_2}\end{pmatrix}$$

The indirect utility function was already calculated:

$$V\left(p_1,p_2,p_3,y\right)=U\left(x\left(p_1,p_2,p_3,y\right)\right)=\frac{y}{p_3}+\frac{p_3^2}{27p_1p_2} \qquad \text{(V)}$$

The Hicksian demand function is the solution to the following expenditure-minimization problem (EMP):

$$\begin{aligned}&\min_{(x_1,x_2,x_3)} p_1x_1+p_2x_2+p_3x_3\\ &s.t.\\ &U\left(x_1,x_2,x_3\right)=\left(x_1x_2\right)^{\frac{1}{3}}+x_3\geq u\\ &x_1,x_2,x_3\geq 0\end{aligned}$$

which is, of course, equivalent to:

$$\begin{aligned}&\max_{(x_1,x_2,x_3)} -\left(p_1x_1+p_2x_2+p_3x_3\right)\\ &s.t.\\ &-U\left(x_1,x_2,x_3\right)=-\left(\left(x_1x_2\right)^{\frac{1}{3}}+x_3\right)\leq -u\\ &x_1,x_2,x_3\geq 0\end{aligned}$$

The Lagrangean:

$$L\left(x_1,x_2,x_3;\lambda,\mu_1,\mu_2,\mu_3\right)=-\left(p_1x_1+p_2x_2+p_3x_3\right)+\lambda\left[-u+\left(\left(x_1x_2\right)^{\frac{1}{3}}+x_3\right)\right]$$
$$+\mu_1\left(x_1-0\right)+\mu_2\left(x_2-0\right)+\mu_3\left(x_3-0\right)$$

The first-order conditions:

Set (I)

$$\frac{\partial L\left(\underline{x};\lambda,\mu_1,\mu_2,\mu_3\right)}{\partial x_1}=0\Leftrightarrow\lambda\frac{\partial U\left(\underline{x}\right)}{\partial x_1}=p_1-\mu_1\Leftrightarrow p_1-\mu_1=\frac{\lambda}{3}\left(x_1x_2\right)^{-\frac{2}{3}}x_2 \qquad (1.1)$$

$$\frac{\partial L\left(\underline{x};\lambda,\mu_1,\mu_2,\mu_3\right)}{\partial x_2}=0\Leftrightarrow\frac{\partial U\left(\underline{x}\right)}{\partial x_2}=p_2-\mu_2\Leftrightarrow p_2-\mu_2=\frac{\lambda}{3}\left(x_1x_2\right)^{-\frac{2}{3}}x_1 \qquad (1.2)$$

$$\frac{\partial L\left(\underline{x};\lambda,\mu_1,\mu_2,\mu_3\right)}{\partial x_3}=0\Leftrightarrow\frac{\partial U\left(\underline{x}\right)}{\partial x_3}=p_3-\mu_3\Leftrightarrow p_3-\mu_3=\lambda \qquad (1.3)$$

Set (II)

$$\left(x_1x_2\right)^{\frac{1}{3}}+x_3\geq u \qquad (2.1)$$

$$\lambda\geq 0 \qquad (2.2)$$

$$\lambda\left[u-\left(\left(x_1x_2\right)^{\frac{1}{3}}+x_3\right)\right]=0 \qquad (2.3)$$

$$x_1\geq 0 \qquad \mu_1\geq 0 \qquad \mu_1x_1=0 \qquad (3.1)$$

$$x_2\geq 0 \qquad \mu_2\geq 0 \qquad \mu_2x_2=0 \qquad (3.2)$$

$$x_3\geq 0 \qquad \mu_3\geq 0 \qquad \mu_3x_3=0 \qquad (3.3)$$

We have the following possible cases:

- (Interior Solution) $x_1,x_2,x_3>0\overset{(3.1),(3.2),(3.3)}{\Rightarrow}\mu_1=\mu_2=\mu_3=0$

By (1.3), since we are given $p_i>0\quad\forall i=1,2,3$, we get $\lambda=p_3>0$

Hence, by (2.3): $u=\left(x_1x_2\right)^{\frac{1}{3}}+x_3$

$$\frac{(1.1)}{(1.2)}\rightarrow\frac{x_2}{x_1}=\frac{p_1}{p_2}$$

From (1.1):

$$\frac{p_1}{p_3}=\frac{1}{3}\left(\frac{x_2}{x_1^2}\right)^{\frac{1}{3}}\Rightarrow x_1=\frac{1}{27}\left(\frac{p_3^3}{p_1^2p_2}\right)$$

Thus, $x_2 = \frac{1}{27}\left(\frac{p_3^3}{p_1 p_2^2}\right)$

Finally: $u = (x_1 x_2)^{\frac{1}{3}} + x_3 \Rightarrow x_3 = u - \frac{p_3^2}{9 p_1 p_2}$

Hence, a candidate solution is: $h^1(p_1, p_2, p_3, u) = \begin{pmatrix} \frac{p_3^3}{27 p_1^2 p_2} \\ \frac{p_3^3}{27 p_1 p_2^2} \\ u - \frac{p_3^2}{9 p_1 p_2} \end{pmatrix}$

- $x_1 = x_2 = 0, \quad x_3 > 0$

Thus, $\mu_3 = 0$ and by (1.3): $\lambda = p_3 > 0$

Hence, by (2.3): $x_3 = u$

From (1.1): $\mu_1 = p_1 > 0$ Similarly, from (1.2): $\mu_2 = p_2 > 0$

Hence, a candidate solution for the EMP is: $h^2(p_1, p_2, p_3, u) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$

- $x_1, x_2 > 0, \quad x_3 = 0$

Thus, $\mu_1 = \mu_2 = 0$ and by (1.3) or (1.2): $\lambda > 0$

Hence, by (2.3): $u = (x_1 x_2)^{\frac{1}{3}}$

$\frac{(1.1)}{(1.2)} \rightarrow \frac{x_2}{x_1} = \frac{p_1}{p_2}$

Hence, $x_1 = \sqrt{\frac{u^3 p_2}{p_1}}$ and $x_2 = \sqrt{\frac{u^3 p_1}{p_2}}$

But from (1.1): $\mu_1 = \frac{3 p_1}{p_3}\left[\sqrt{\frac{u^3 p_2}{p_1}}\sqrt{\frac{u^3 p_1}{p_2}}\right]^{-\frac{2}{3}}\sqrt{\frac{u^3 p_1}{p_2}} = \sqrt{\frac{u p_1}{p_2}} > 0$. This result clearly violates our earlier premise that $\mu_1 = 0$. Therefore, this case cannot yield a candidate solution[2].

[2] One would reach the same conclusion by solving instead for noticing instead for μ_2 from (1.2).

Note that we don't need to check the sub-cases of each of the cases (a) $x_1 = x_2 = 0, \quad x_3 > 0$ and (b) $x_1, x_2 > 0, \quad x_3 = 0$ where one of x_1, x_2 is strictly positive while the other being zero for exactly the same reason as in the UMP.

It is easy to verify that, from the two candidate solutions that we found above, the first one yields the lowest level of expenditure. Specifically,

$$
\begin{aligned}
& m\left(h^1\left(p_1, p_2, p_3, u\right)\right) \\
& = p_1 h_1^1 + p_2 h_2^1 + p_3 h_3^1 \\
& = p_3 u - \frac{p_3^2}{27 p_1 p_2} \\
& < p_3 u \\
& = p_1 h_1^2 + p_2 h_2^2 + p_3 h_3^2 \\
& = m\left(h^2\left(p_1, p_2, p_3, u\right)\right)
\end{aligned}
$$

Finally, the Hicksian demand is given:

$$
h\left(p_1, p_2, p_3, u\right) = \begin{pmatrix} \dfrac{p_3^3}{27 p_1^2 p_2} \\ \dfrac{p_3^3}{27 p_1 p_2^2} \\ u - \dfrac{p_3^2}{9 p_1 p_2} \end{pmatrix}
$$

The expenditure function was already calculated:

$$
\begin{aligned}
& m\left(p_1, p_2, p_3, u\right) = m\left(h\left(p_1, p_2, p_3, u\right)\right) \\
& = p_1 h_1^1\left(p_1, p_2, p_3, u\right) + p_2 h_2^1\left(p_1, p_2, p_3, u\right) + p_3 h_3^1\left(p_1, p_2, p_3, u\right) \qquad \text{(E)} \\
& = p_3 u - \frac{p_3^3}{27 p_1 p_2}
\end{aligned}
$$

At the solution points for the UMP and EMP, we can interchange using the following identities:

1. $x(p,y) = h(p,u) = h(p, V(p,y))$

To see this, consider the EMP when the agent is required to achieve at least the utility level corresponding to his indirect utility level from (V) when prices and income are given by (p,y).

We get from (V) for $u = V(p_1,p_2,p_3,y)$

$$u = \frac{y}{p_3} + \frac{p_3^2}{27p_1p_2}$$

For this required reservation level of utility, the solution to the EMP will be given:

$$h(p_1,p_2,p_3,u) = \begin{pmatrix} \frac{p_3^3}{27p_1^2p_2} \\ \frac{p_3^3}{27p_1p_2^2} \\ u - \frac{p_3^2}{9p_1p_2} \end{pmatrix} = \begin{pmatrix} \frac{p_3^3}{27p_1^2p_2} \\ \frac{p_3^3}{27p_1p_2^2} \\ \left(\frac{y}{p_3} + \frac{p_3^2}{27p_1p_2}\right) - \frac{p_3^2}{9p_1p_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p_3^3}{27p_1^2p_2} \\ \frac{p_3^3}{27p_1p_2^2} \\ \frac{y}{p_3} - \frac{2p_3^2}{27p_1p_2} \end{pmatrix} = x(p_1,p_2,p_3,y)$$

2. $x(p, m(p,u)) = h(p,u)$

To see this, consider the UMP when the agent's income is the minimum amount he needs in his EMP in order to achieve at least a utility level *u*.
We get from (E) that the minimum amount of money required to achieve a reservation level of utility equal to *u* is given:

$$m(p,u) = p_3u - \frac{p_3^2}{27p_1p_2}$$

When this is the amount of income available for the UMP, the solution to the UMP will be given:

$$x\left(p_1,p_2,p_3,m\left(p,u\right)\right)=\begin{pmatrix}\dfrac{p_3^3}{27p_1^2p_2}\\\dfrac{p_3^3}{27p_1p_2^2}\\\dfrac{e\left(p,u\right)}{p_3}-\dfrac{2p_3^2}{27p_1p_2}\end{pmatrix}=\begin{pmatrix}\dfrac{p_3^3}{27p_1^2p_2}\\\dfrac{p_3^3}{27p_1p_2^2}\\\left(u-\dfrac{p_3^2}{27p_1p_2}\right)-\dfrac{2p_3^2}{27p_1p_2}\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{p_3^3}{27p_1^2p_2}\\\dfrac{p_3^3}{27p_1p_2^2}\\u-\dfrac{p_3^2}{9p_1p_2}\end{pmatrix}=h\left(p_1,p_2,p_3,u\right)$$

3. $m\left(p,V\left(p,y\right)\right)=y$

To see this, consider the EMP when the agent is required to achieve at least the utility level corresponding to his indirect utility level from (V) when prices and income are given by $\left(p,y\right)$.

We get from (V) for $u=V\left(p_1,p_2,p_3,y\right)$: $u=\dfrac{y}{p_3}+\dfrac{p_3^2}{27p_1p_2}$

Now consider the EMP when the agent is required to achieve this as the reservation level of his utility. From (E), the minimum amount of money required to achieve this reservation level of utility is given:

$$m\left(p,u\right)=p_3\left(\frac{y}{p_3}+\frac{p_3^2}{27p_1p_2}\right)-\frac{p_3^3}{27p_1p_2}=y$$

4. $V\left(p,m\left(p,u\right)\right)=u$

Consider the UMP when the agent's income is the minimum amount he needs in his EMP in order to achieve at least a utility level *u*.

We get from (E) that the minimum amount of money required to achieve a reservation level of utility equal to *u* is given:

$$m\left(p,u\right)=p_3u-\frac{p_3^3}{27p_1p_2}$$

When this is the amount of income available for the UMP, the solution to the UMP will yield a utility level given by (V):

$$V\left(p,m\left(p,u\right)\right)=\frac{m\left(p,u\right)}{p_3}+\frac{p_3^2}{27p_1p_2}=\frac{1}{p_3}\left(p_3u-\frac{p_3^3}{27p_1p_2}\right)+\frac{p_3^2}{27p_1p_2}=u$$

(B) This involves only algebra and I will leave it to you to verify.

(C) This involves only algebra and I will leave it to you to verify.

(D) Consider the following (unconstrained) minimization problem:

$$\min_{p_1,p_2,p_3} V\left(p_1,p_2,p_3,\sum_{i=1}^{3}p_ix_i\right)=\frac{\sum_{i=1}^{3}p_ix_i}{p_3}+\frac{p_3^2}{27p_1p_2}$$

The first order conditions:

$$\frac{\partial V\left(p_1,p_2,p_3,\sum_{i=1}^{3}p_ix_i\right)}{\partial p_1}=0\Leftrightarrow\frac{x_1}{p_3}-\frac{p_3^2}{27p_1^2p_2}=0\Leftrightarrow x_1=\frac{p_3^3}{27p_1^2p_2}\qquad\text{(D.1)}$$

$$\frac{\partial V\left(p_1,p_2,p_3,\sum_{i=1}^{3}p_ix_i\right)}{\partial p_2}=0\Leftrightarrow\frac{x_2}{p_3}-\frac{p_3^2}{27p_2^2p_1}=0\Leftrightarrow x_2=\frac{p_3^3}{27p_2^2p_1}\qquad\text{(D.2)}$$

$$\frac{\partial V\left(p_1,p_2,p_3,\sum_{i=1}^{3}p_ix_i\right)}{\partial p_3}=0\Leftrightarrow\frac{x_3p_3-\left(\sum_{i=1}^{3}p_ix_i\right)}{p_3^2}-\frac{2p_3}{27p_1p_2}=0\Leftrightarrow p_1x_1+p_2x_2=\frac{2p_3^3}{27p_1p_2}$$

(which holds whenever (D.1) and (D.2) do)

The minimized expenditure (i.e. the value function) of the problem is found by plugging the solution into the objective.

$$\min_{p_1,p_2,p_3} V\left(p_1,p_2,p_3,\sum_{i=1}^{3} p_i x_i\right)$$

$$= \frac{\sum_{i=1}^{3} p_i x_i}{p_3} + \frac{p_3^2}{27 p_1 p_2}$$

$$= \frac{p_1}{p_3}x_1 + \frac{p_2}{p_3}x_2 + x_3 + \frac{p_3^2}{27 p_1 p_2}$$

$$= x_3 + \frac{p_1}{p_3}\left(\frac{p_3^3}{27 p_1^2 p_2}\right) + \frac{p_2}{p_3}\left(\frac{p_3^3}{27 p_1 p_2^2}\right) + \frac{p_3^2}{27 p_1 p_2}$$

$$= x_3 + \frac{p_3^2}{27 p_1 p_2} + \frac{p_3^2}{27 p_1 p_2} + \frac{p_3^2}{27 p_1 p_2}$$

$$= x_3 + \frac{p_3^2}{9 p_1 p_2}$$

$$= x_3 + \left(\frac{p_3}{3\sqrt[3]{p_1^2 p_2}}\right)\left(\frac{p_3}{3\sqrt[3]{p_1 p_2^2}}\right)$$

$$= x_3 + \left[\left(\frac{p_3^3}{27 p_1^2 p_2}\right)\left(\frac{p_3^3}{27 p_1 p_2^2}\right)\right]^{\frac{1}{3}}$$

$$= x_3 + \left(x_1 x_2\right)^{\frac{1}{3}}$$

$$= U\left(x_1,x_2,x_3\right)$$

Consider now the following (constrained) minimization problem:

$$\min_{(p_1,p_2,p_3)} u$$

$$s.t.$$

$$e(p,u) = p_3 u - \frac{p_3^3}{27 p_1 p_2} \geq p_1 x_1 + p_2 x_2 + p_3 x_3$$

$$p_1, p_2, p_3 \geq 0$$

which is, of course, equivalent to:

$$\max_{(p_1,p_2,p_3)} -u$$
$$s.t.$$
$$-\left(p_3u-\frac{p_3^3}{27p_1p_2}\right)\le -p_1x_1-p_2x_2-p_3x_3$$
$$p_1,p_2,p_3\ge 0$$

Note first that we are looking for a solution to this problem, which ought to be valid *for all* $p_1,p_2,p_3\ge 0$. In this respect, this problem differs substantially from the constrained optimization problems that we have seen thus far in that we are no longer allowed to consider cases for our choice variables p_1,p_2,p_3. In other words, coming up with a candidate solution that requires, for example $p_1,p_2,p_3>0$ but is not valid when $p_1,p_2>0,\quad p_3=0$ will not do here. We need to fnd a solution that will operate as an identity in the sense that will be valid across all interior and corner cases as long as $p_1,p_2,p_3\ge 0$.

The Lagrangean:

$$L(p_1,p_2,p_3;\lambda,\mu_1,\mu_2,\mu_3)=-u+\lambda\left[-p_1x_1-p_2x_2-p_3x_3+\left(p_3u-\frac{p_3^3}{27p_1p_2}\right)\right]$$
$$+\mu_1(p_1-0)+\mu_2(p_2-0)+\mu_3(p_3-0)$$

The first-order conditions:

Set (I)

$$\frac{\partial L(p;\lambda,\mu_1,\mu_2,\mu_3)}{\partial p_1}=0\Leftrightarrow\frac{\lambda p_3^3}{27p_1^2p_2}=\lambda x_1-\mu_1 \qquad \text{(D.I)}$$

$$\frac{\partial L(p;\lambda,\mu_1,\mu_2,\mu_3)}{\partial p_2}=0\Leftrightarrow\frac{\lambda p_3^3}{27p_1p_2^2}=\lambda x_2-\mu_2 \qquad \text{(D.II)}$$

$$\frac{\partial L(p;\lambda,\mu_1,\mu_2,\mu_3)}{\partial p_3}=0\Leftrightarrow\lambda u-\frac{3\lambda p_3^2}{27p_1p_2}=\lambda x_3-\mu_3 \qquad \text{(D.III)}$$

<u>Set (II)</u>

$$\left(p_3 u - \frac{p_3^3}{27 p_1 p_2}\right) \geq p_1 x_1 + p_2 x_2 + p_3 x_3 \qquad \text{(D.2.1)}$$

$$\lambda \geq 0 \qquad \text{(D.2.2)}$$

$$\lambda\left[p_1 x_1 + p_2 x_2 + p_3 x_3 - \left(p_3 u - \frac{p_3^3}{27 p_1 p_2}\right)_3\right] = 0 \qquad \text{(D.2.3)}$$

$$p_1 \geq 0 \qquad \mu_1 \geq 0 \qquad \mu_1 p_1 = 0 \qquad \text{(D.3.1)}$$

$$p_2 \geq 0 \qquad \mu_2 \geq 0 \qquad \mu_2 p_2 = 0 \qquad \text{(D.3.2)}$$

$$p_3 \geq 0 \qquad \mu_3 \geq 0 \qquad \mu_3 p_3 = 0 \qquad \text{(D.3.3)}$$

Again, we need a solution, which is to work for any $p_1, p_2, p_3 \geq 0$. Therefore, it ought to work also when we have an interior solution i.e. $p_1, p_2, p_3 > 0$

But $p_1, p_2, p_3 > 0 \overset{(D.3.1),(D.3.2),(D.3.3)}{\Rightarrow} \mu_1 = \mu_2 = \mu_3 = 0$.

Therefore, in our solution, we require: $\mu_1 = \mu_2 = \mu_3 = 0$

Similarly, we must have $\lambda > 0$ in our solution.
For if we don't, we leave p_1, p_2, p_3 completely free relative to x_1, x_2 and this will create problems in the case where $p_1, p_2 > 0 \quad p_3 = 0$.
Given that $\mu_1 = \mu_2 = \mu_3 = 0$, observe that $\lambda = 0$ has all equations but (D.2.1) satisfied trivially <u>*irrespectively*</u> of the values of p_1, p_2, p_3 and x_1, x_2, x_3. To satisfy all conditions, we only need to worry about (D.2.1).

For $p_1, p_2 > 0 \quad p_3 = 0$, this gives: $\qquad p_1 x_1 + p_2 x_2 \leq 0$
which cannot be satisfied for any $p_1, p_2 > 0$ unless $x_1 = x_2 = 0$. But recall that the expenditure function $m(p,u)$ was derived as the value function of the EMP which does admitted only a strictly interior solution where $x_1, x_2, x_3 > 0$. Hence, we cannot have $x_1 = x_2 = 0$ and consequently we cannot have $\lambda = 0$.

- Take therefore $\mu_1 = \mu_2 = \mu_3 = 0$ and $\lambda > 0$

$$\frac{(D.I)}{(D.II)} \to \frac{x_2}{x_1} = \frac{p_1}{p_2}$$

From (D.2.3) we get:

$$p_3 u - \frac{p_3^3}{27 p_1 p_2} = p_1 x_1 + p_2 x_2 + p_3 x_3 \Rightarrow$$

$$u = \frac{p_1}{p_3} x_1 + \frac{p_2}{p_3} x_2 + x_3 + \frac{p_3^3}{27 p_1 p_2}$$

Proceed now as in the derivation of the result for the previous minimization problem (see pp. 11) to derive the functional form of *u*:

$$u = (x_1 x_2)^{\frac{1}{3}} + x_3$$

(D) This involves only algebra (albeit quite tedious in showing concavity of $m(p,u)$ and convexity of $V(p,y)$ since one needs to establish the definiteness of the 4*x*4 Hessian matrices) and I will leave it to you to verify.

(E) This involves only algebra and I will leave it to you to verify.

(F) This involves only algebra and I will leave it to you to verify.
Note, however, that there are some typos in the text of this part of the problem set.
The given relations should read:

(pp. 1)

$$\frac{\partial x_i(p,y)}{\partial p_j} = \left.\frac{\partial h_i(p,u)}{\partial p_j}\right|_{u=V(p,y)} + \left.\frac{\partial h_i(p,u)}{\partial u}\right|_{u=V(p,y)} \frac{\partial V(p,y)}{\partial p_j}$$

$$\frac{\partial x_i(p,y)}{\partial p_j} = \left.\frac{\partial h_i(p,u)}{\partial u}\right|_{u=V(p,y)} \frac{\partial V(p,y)}{\partial y}$$

(pp.2)

$$\frac{\partial x_i(p,y)}{\partial p_j} = \left.\frac{\partial h_i(p,u)}{\partial p_j}\right|_{u=V(p,y)} + \frac{\frac{\partial x_i(p,y)}{\partial y}}{\frac{\partial V(p,y)}{\partial y}} \frac{\partial V(p,y)}{\partial p_j}$$

$$\frac{\partial x_i(p,y)}{\partial p_j} = \left.\frac{\partial h_i(p,u)}{\partial p_j}\right|_{u=V(p,y)} - x_i(p,y)\frac{\partial x_i(p,y)}{\partial y}$$

$$\frac{\partial x_i(p,y)}{\partial p_j} = \left.\frac{\partial^2 m(p,u)}{\partial p_i \partial p_j}\right|_{u=V(p,y)} - x_i(p,y)\frac{\partial x_i(p,y)}{\partial y}$$

Problem 2

(A) Consider the profit-maximization problem (PMP)[3]:

$$\max_{(z_1,z_2,z_3)} pz = \sum_{i=1}^{3} p_i z_i$$

$$s.t.$$

$$z_1 \le \left(z_2 z_3^3\right)^{\frac{1}{5}} \quad z_2, z_3 \le 0$$

The Lagrangean:

$$L(z_1,z_2,z_3;\lambda,\mu_2,\mu_3) = \sum_{i=1}^{3} p_i z_i + \lambda\left[\left(z_2 z_3^3\right)^{\frac{1}{5}} - z_1\right] + \mu_2(0-z_2) + \mu_3(0-z_3)$$

The first-order conditions:

Set (I)

$$\frac{\partial L(z;\lambda,\mu_2,\mu_3)}{\partial z_1} = 0 \Leftrightarrow \lambda = p_1 \qquad (1.1)$$

$$\frac{\partial L(z;\lambda,\mu_2,\mu_3)}{\partial z_2} = 0 \Leftrightarrow = p_2 + \frac{\lambda}{5}\left(z_2 z_3^3\right)^{-\frac{4}{5}} z_3^3 = \mu_2 \qquad (1.2)$$

$$\frac{\partial L(z;\lambda,\mu_2,\mu_3)}{\partial z_3} = 0 \Leftrightarrow p_3 + \frac{3\lambda}{5}\left(z_2 z_3^3\right)^{-\frac{4}{5}} z_2 z_3^2 = \mu_3 \qquad (1.3)$$

---

[3] Note that only the two input variables z_2, z_3 are constrained (to be non-positive) in this problem.

Set (II)

$$z_1 \le \left(z_2 z_3^3\right)^{\frac{1}{5}} \qquad (2.1)$$
$$\lambda \ge 0 \qquad (2.2)$$
$$\lambda\left[z_1 - \left(z_2 z_3^3\right)^{\frac{1}{5}}\right] = 0 \qquad (2.3)$$

$$z_2 \le 0 \qquad \mu_2 \ge 0 \qquad \mu_2 z_2 = 0 \qquad (3.1)$$

$$z_3 \le 0 \qquad \mu_3 \ge 0 \qquad \mu_3 z_3 = 0 \qquad (3.2)$$

From (1.1), it is clear that $\lambda > 0$. Therefore, from (2.3): $z_1 = \left(z_2 z_3^3\right)^{\frac{1}{5}}$

Note that an obvious point that satisfies all of our first order conditions is the point $z = (z_1, z_2, z_3) = (0,0,0)$ with $\mu_2 = p_2 > 0,\ \ \mu_3 = p_3 > 0$. However, not producing nothing at all results in zero profit. Consequently, *any* other vector *z* that generates a positive amount of profit would be preferred to the zero-point and, thus, this point cannot be optimal.
For exactly the same reason, any vector *z* involving $z_i = 0$ for either of $i = 2,3$ cannot be optimal as it gives zero output (i.e. zero revenues from sales and non-positive profit).

We have only the following case to consider:

- (Interior Solution) $\quad z_2, z_3 < 0 \overset{(3.1),(3.2)}{\Rightarrow} \mu_2 = \mu_3 = 0$

$$\frac{(1.2)}{(1.3)} \to \frac{z_3}{3z_2} = \frac{p_2}{p_3}$$

From (1.2) and (1.1):

$$\frac{5p_2}{p_1} = \left(z_2 z_3^3\right)^{-\frac{4}{5}} z_3^3 = \left(\frac{p_3}{3p_2} z_3^4\right)^{-\frac{4}{5}} z_3^3 \Rightarrow \quad z_3 = \left(\frac{p_3}{3p_2}\right)^{-4}\left(\frac{p_1}{5p_2}\right)^5 = \frac{3^4 p_1^5}{5^5 p_3^4 p_2}$$

And

$$z_2 = \frac{3^3 p_1^5}{5^5 p_3^3 p_2^2} \qquad z_1 = \left[\left(\frac{3^3 p_1^5}{5^5 p_3^3 p_2^2}\right)\left(\frac{3^4 p_1^5}{5^5 p_3^4 p_2}\right)^3\right]^{\frac{1}{5}} = \left(\frac{3^{15} p_1^{20}}{5^{20} p_3^{15} p_2^5}\right)^{\frac{1}{5}} = \frac{3^3 p_1^4}{5^4 p_3^3 p_2}$$

Therefore

$$z\left(p_1,p_2,p_3\right)=\left(z_1\left(p\right),z_2\left(p\right),z_3\left(p\right)\right)=\left(\frac{3^3}{5^4}\frac{p_1^4}{p_3^3p_2},\frac{3^3}{5^5}\frac{p_1^5}{p_3^3p_2^2},\frac{3^4}{5^5}\frac{p_1^5}{p_3^4p_2}\right)$$

The profit function is given:

$$\pi\left(p\right)=\sum_{i=1}^{3}p_iz_i\left(p\right)=\frac{3^3}{5^4}\frac{p_1^5}{p_3^3p_2}-\frac{3^3}{5^4}\frac{p_1^5}{p_3^3p_2}-\frac{3^4}{5^4}\frac{p_1^5}{p_3^3p_2}=\frac{3^4}{5^4}\frac{p_1^5}{p_3^3p_2}$$

I will leave to you to verify that $z_i\left(p\right)=\frac{\partial\pi\left(p\right)}{\partial p_i}\quad\forall i\in\{1,2,3\}$ as well as that $\pi\left(p\right)$ is convex in p (the latter exercise will be rather tedious as you need to determine the definiteness of a $3x3$ Hessian matrix).

(B) The ***free disposal property*** can be written as follows:

$$z=\left(z_1,z_2,z_3\right)\in T\Rightarrow\left\{\tilde{z}\in R^3:\tilde{z}_1\le z_1\wedge\tilde{z}_2\le z_2\wedge\tilde{z}_3\le z_3\right\}\subseteq T$$

We are given that the production possibility set (PPS): $T=\left\{z\in R^3:pz\le\pi\left(p\right)\quad\forall p>>0\right\}$ is closed.

This means that it ought to include its boundary: $\partial T=\left\{z\in R^3:pz=\pi\left(p\right)\quad\forall p>>0\right\}$

Consider this boundary set. It consists of those points z that satisfy the relation:

$$pz=\sum_{i=1}^{3}p_iz_i=\pi\left(p\right)\quad\forall p>>0\qquad\text{(I).}$$

Note, however, that since this relation is to hold for all strictly positive price vectors, it is really an identity relation with respect to *p*. Therefore, if we consider the gradient of each side of (I) with respect to *p*, the two sides should agree[4].
i.e.

$$pz = \sum_{i=1}^{3} p_i z_i = \pi(p) \qquad \forall p >> 0 \Rightarrow$$

$$\nabla_p pz = \nabla_p \pi(p) \qquad \forall p >> 0 \Rightarrow$$

$$\left(\frac{\partial \sum_{i=1}^{3} p_i z_i}{\partial p_i} \right)_{i=1}^{3} = \left(\frac{\partial \pi(p)}{\partial p_i} \right)_{i=1}^{3} \qquad \forall p >> 0 \Rightarrow$$

$$\frac{\partial \sum_{i=1}^{3} p_i z_i}{\partial p_i} = \frac{\partial \pi(p)}{\partial p_i} \quad \forall i \in \{1,2,3\} \qquad \forall p >> 0$$

In other words, we get: $z_i = \frac{\partial \pi(p)}{\partial p_i} \quad \forall i \in \{1,2,3\}$ and this ought to be a *__necessary__* condition, if (I) is to hold as an identity for all price vectors $p >> 0$.

Regarding the given profit function, we have:

$$z_1 = \frac{2p_1}{p_3} \quad z_2 = \frac{4p_2}{p_3} \quad z_3 = -\frac{\left(p_1^2 + 2p_2^2\right)}{p_3^2} \qquad \text{(I.1)}$$

Note that (I.1) is required to hold as a system of equations, for all $p >> 0$. Therefore, it defines the following inter-relation between the elements of the supply vector z:

$$z_1, z_2 > 0 \quad \wedge \quad z_3 = -\left(\frac{z_1^2}{4} + \frac{z_2^2}{8} \right) \Leftrightarrow 8z_3 = -\left(2z_1^2 + z_2^2\right)$$

---

[4] Essentially, I am using the following argument. Let $f(x) = g(x) \quad \forall x \in domf \cap domg$ (I) Then, clearly $f\big|_{domf \cap domg} \equiv g\big|_{domf \cap domg}$ and, consequently, $f'\big|_{domf \cap domg} \equiv g'\big|_{domf \cap domg}$. In other words, (I) implies $f'(x) = g'(x) \quad \forall x \in domf \cap domg$.

We are now in a position of being able to write the boundary set for the PPS under examination.

$$\partial T = \left\{ z \in R^3 : pz = \pi(p) \quad \forall p >> 0 \right\}$$
$$= \left\{ z \in R^3 : \ z_1 > 0, \ z_2 > 0, \ 8z_3 = -\left(2z_1^2 + z_2^2\right) \right\}$$
$$= \left\{ \left(z_1, z_2, -\left(2z_1^2 + z_2^2\right) \right) : z_1, z_2 \in R_+^* \right\}$$

Recall now that the PPS has to satisfy the free disposal property. Hence, it will consist of the boundary set ∂T and all vectors that lie to the south-west of any given point on this boundary. i.e.

$T = \left\{ (z_1, z_2, z_3) \in R^3 : \ z_1 > 0, \ z_2 > 0, \ 8z_3 \leq -\left(2z_1^2 + z_2^2\right) \right\}$ or

$T = \left\{ (z_1, z_2, z_3) \in R^3 : \ z_1 > 0, \ z_2 > 0, \ 2z_1^2 + z_2^2 \leq -8z_3 \right\}$

<u>Notes:</u>

- To complete the derivation of the production possibility set *T*, you should verify that the set given here is convex. This is quite straight forward as it suffices to show that the boundary set is concave.
- The problem also asks for us to verify that $\pi(p) = \frac{\left(p_1^2 + 2p_2^2\right)}{p_3}$ is indeed the profit function associated with this production possibility set. This calls merely for you to repeat part (A) of the problem using $T = \left\{ (z_1, z_2, z_3) \in R^3 : \ z_1 > 0, \ z_2 > 0, \ 2z_1^2 + z_2^2 \leq -8z_3 \right\}$ as your production possibility set. I will, thus, leave it to you to show that this actually works[5].

---

[5] It would be again clear from the FOC's on the Lagrangean that $\lambda > 0$. Thus: $2z_1^2 + z_2^2 = -8z_3$
An obvious point, again, that satisfies all of the FOC's is the point $z = (z_1, z_2, z_3) = (0,0,0)$ with $\mu_1 = p_1 > 0, \ \ \mu_2 = p_2 > 0$. It is ruled out, though, for exactly the same reasoning as in part (A). However, here we cannot rule out a priori any vector *z* involving $z_i = 0$ for either of $i = 1,2$ as long as it is not for both, since such a vector does not necessarily give zero total output (as was the case in part (A)). For $z_1 = 0, z_2 > 0$, the candidate solution is $z^1 = (z_1, z_2, z_3) = \left(0, \frac{4p_2}{p_3}, -\frac{2p_2^2}{p_3^2} \right)$. Similarly, $z_1 > 0, z_2 = 0$ gives $z^2 = \left(\frac{2p_1}{p_3}, 0, -\frac{p_1^2}{p_3^2} \right)$. The former gives a profit of $\frac{2p_2^2}{p_3}$ while the latter gives $\frac{p_1^2}{p_3}$. Neither beats the interior solution's profit $\frac{p_1^2 + 2p_2^2}{p_3}$.