PROBLEM SET I
(Suggested Solutions)

1.

a) Consider the following: $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

The quadratic form $x^T Ax$ is the required one in matrix form.

Similarly, for the following parts:

b) $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $A = \begin{pmatrix} 5 & 0 \\ -10 & -1 \end{pmatrix}$

c) $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $A = \begin{pmatrix} 1 & 0 & -6 \\ 4 & 2 & 0 \\ 0 & 8 & 3 \end{pmatrix}$

2.

Recall that, for any square matrix $\underset{n\times n}{A} = \left[a_{ij} \right]_{i=1,\, j=1}^{n}$, if $\underset{(n-1)\times(n-1)}{A_{ij}}$ the sub-matrix that is left after we delete the *i*th row and *j*th column of *A*, the scalar $M_{ij} = \det\left(A_{ij}\right)$ is defined as the (*i*,*j*)th **minor** of *A*.
The *kxk* sub-matrix of *A* that is obtained when the first *k* rows and columns of *A* are retained is called the ***k*th-order leading principal minor** of *A*:

$$M_k = \begin{vmatrix} a_{11} & a_{12} & . & a_{1k} \\ . & & & . \\ . & & & . \\ a_{k1} & a_{k2} & . & a_{kk} \end{vmatrix} : k = 1,2,...,n$$

Let $\pi = \left(\pi_1, \pi_2, .., \pi_n\right)$ a permutation of the integers $\{1,2,...,n\}$ and Π the set of all permutations of the integers $\{1,2,...,n\}$. Denote by A^{π} the symmetric *nxn* matrix obtained by applying the permutation π to both the rows and the columns of *A*:

$$A^{\pi} = \begin{bmatrix} a_{\pi_1\pi_1} & a_{\pi_1\pi_2} & . & a_{\pi_1\pi_n} \\ . & & & . \\ . & & & . \\ a_{\pi_n\pi_1} & a_{\pi_n\pi_2} & . & a_{\pi_n\pi_n} \end{bmatrix}$$

For $k \in \{1,2,..,n\}$, let A_k^{π} denote the *kxk* submatrix of A^{π} obtained by retaining only the first *k* rows and columns:

$$A_k^{\pi} = \begin{bmatrix} a_{\pi_1\pi_1} & a_{\pi_1\pi_2} & . & a_{\pi_1\pi_k} \\ . & & & . \\ . & & & . \\ a_{\pi_k\pi_1} & a_{\pi_k\pi_2} & . & a_{\pi_k\pi_k} \end{bmatrix}$$

For $k \in \{1,2,..,n\}$ and $\pi \in \Pi$, A_k^{π} is a **kth-order principal minor** of *A*.

A real, symmetric matrix $\underset{n\times n}{A} = \left[a_{ij}\right]_{i=1,\,j=1}^{n}$ is **positive definite** if and only if all of its leading principal minors are positive:

$$\begin{vmatrix} a_{11} & a_{12} & . & a_{1k} \\ . & & & . \\ . & & & . \\ a_{k1} & a_{k2} & . & a_{kk} \end{vmatrix} > 0 : k = 1,2,...,n$$

On the other hand, a real symmetric matrix $\underset{n\times n}{A} = \left[a_{ij}\right]_{i=1,\,j=1}^{n}$ is **positive semi-definite** if and only if $\left|A_k^{\pi}\right| \geq 0$ for all $k = 1,2,...,n$ and for all $\pi \in \Pi$.

A real symmetric matrix $\underset{n\times n}{A} = \left[a_{ij}\right]_{i=1,\,j=1}^{n}$ is **negative definite** if and only if the sequence of its *n* leading principal minors alternate in sign starting with the first one being negative (or, equivalently, *-A* is positive definite):

$$\left(-1\right)^k \begin{vmatrix} a_{11} & a_{12} & . & a_{1k} \\ . & & & . \\ . & & & . \\ a_{k1} & a_{k2} & . & a_{kk} \end{vmatrix} > 0 : k = 1,2,...,n$$

On the other hand, a real symmetric matrix $\underset{n\times n}{A} = \left[a_{ij}\right]_{i=1,\,j=1}^{n}$ is **negative semi-definite** if and only if $\left(-1\right)^k \left|A_k^{\pi}\right| \geq 0$ for all $k = 1,2,...,n$ and for all $\pi \in \Pi$.

a) Consider the leading principal minors of the given matrix:

$$M_1 = -3 < 0$$

$$M_2 = \begin{vmatrix} -3 & 4 \\ 4 & -6 \end{vmatrix} = 18 - 16 = 2 > 0$$

The given matrix is negative definite.

b) $M_1 = 2 > 0$

$$M_2 = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 2 - 1 = 1 > 0$$

The given matrix is positive definite.

c) $M_1 = 1 > 0$

$$M_2 = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$$

$$M_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{vmatrix} = 1x\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2x\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix} + 0 = (24 - 25) - 2x(12 - 0) = -25 < 0$$

The given matrix is neither positive nor negative (semi-)definite. Its definiteness cannot be determined (indefinite).

d) $M_1 = 1 > 0$

$$M_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 > 0$$

$$M_3 = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{vmatrix} = 1x\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + 0 + 3x\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = (8 - 0) + 3x(0 - 6) = 8 - 18 = -10 < 0$$

Note that we need not proceed to calculate the fourth principal minor. Since one of the principal minors came out negative, the matrix cannot be positive definite. Hence, the given matrix is indefinite.

3.

An *nxn* matrix $\underset{n\times n}{A} = \left[a_{ij} \right]_{i=1,j=1}^{n}$ will have as many *k*th-order principal minors as there are permutations π of *k* integers out of the total of *n*. Hence, it will have $\frac{n!}{k!(n-k)!}$ *k*th-order principal minors for $k = 1, 2, ..., n$.

4.

Consider some vector $x \in R^N$: $x = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_N \end{pmatrix}$ and some *NxN* square matrix $\underset{n \times n}{A} = \left[a_{ml} \right]_{m=1,l=1}^{n}$.

The quadratic form $Q = x^T Ax$ can be written analytically as follows:

$$Q = x^T Ax = \begin{pmatrix} x_1 & x_2 & . & . & x_N \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & . & a_{1N} \\ . & & & . \\ . & & & . \\ a_{N1} & a_{N2} & . & a_{NN} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_N \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{21}x_2 + .. + a_{N1}x_N & a_{12}x_1 + a_{22}x_2 + .. + a_{N2}x_N & . & . & a_{1N}x_1 + a_{2N}x_2 + .. + a_{NN}x_N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_N \end{pmatrix}$$

$$= \left(a_{11}x_1^2 + a_{21}x_1x_2 + .. + a_{N1}x_1x_N \right) + \left(a_{12}x_1x_2 + a_{22}x_2^2 + .. + a_{N2}x_2x_N \right)$$
$$+ ... + \left(a_{1N}x_1x_N + a_{2N}x_2x_N + .. + a_{NN}x_N^2 \right)$$

Let us evaluate now this quadratic form along the coordinates of R^N.

Consider the vector corresponding to the dimension along the first coordinate $x^1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$.

We get

$$Q = x^{1T} Ax^1 = \begin{pmatrix} 1 & 0 & . & . & 0 \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & . & a_{1N} \\ . & & & . \\ . & & & . \\ a_{N1} & a_{N2} & . & a_{NN} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ . \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11}, & a_{12}, & . & . & a_{1N} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ . \\ 0 \end{pmatrix} = a_{11}$$

Similarly for the vector corresponding to the dimension along the *m*th coordinate

$x^m = \begin{pmatrix} 0 \\ 0 \\ . \\ 1 \\ . \\ 0 \end{pmatrix}$ (this vector has an entry of 1 at the *m*th row and zeros everywhere else)

We get

$$Q = x^{mT} A x^{m} = \begin{pmatrix} 0 & 0 & . & 1 & . & 0 \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & . & a_{1N} \\ . & & & . \\ . & & & . \\ a_{N1} & a_{N2} & . & a_{NN} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ . \\ 1 \\ . \\ 0 \end{pmatrix} = \begin{pmatrix} a_{m1}, & a_{m2}, & . & . & a_{mN} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ . \\ 1 \\ . \\ 0 \end{pmatrix} = a_{mm}$$

We are given that the quadratic form Q is positive definite. Hence, we ought to have $Q > 0$ for <u>*any*</u> $x \in R^N$. Thus, it ought to be $Q > 0$ also for any $x^m \in R^N : m = 1,2,..,N$
This gives $a_{mm} > 0 : m = 1,2,..,N$. Hence, we establish that
$Q > 0 \Rightarrow a_{mm} > 0 : \forall m = 1,2,...,N$ i.e. a necessary condition for the generic square matrix A to be positive definite is that all its diagonal entries a_{mm} to be positive.

In the case where the quadratic form Q is positive semi-definite, we have $Q \geq 0$ for <u>*any*</u> $x \in R^N$. Thus, it ought to be $Q \geq 0$ also for any $x^m \in R^N : m = 1,2,..,N$
This gives $a_{mm} \geq 0 : m = 1,2,..,N$. Hence, we establish that
$Q \geq 0 \Rightarrow a_{mm} \geq 0 : \forall m = 1,2,...,N$ i.e. a necessary condition for the generic square matrix A to be positive semi-definite is that all its diagonal entries a_{mm} to be non-negative.

One can follow the above proof for the case where the matrix A is negative (semi-) definite. It is easy to establish that: $Q < 0 \Rightarrow a_{mm} < 0 : \forall m = 1,2,...,N$ and $Q \leq 0 \Rightarrow a_{mm} \leq 0 : \forall m = 1,2,...,N$. Hence, a necessary condition is that all the diagonal entries are negative (non-positive) respectively.

To show that the above conditions are not sufficient, consider the matrix in part (c) of problem 1 above. For this matrix, all of its diagonal entries are positive, yet it is not positive definite.

5.

Let $x^1, x^2 \in R^m$ and $\lambda \in R$.

From the definition of $h: R^m \to R$ and $f: R^n \to R$ being concave, we get:

$$h\left(\left(\lambda x^1 + (1-\lambda) x^2\right)\right) =$$

$$f\left(A\left(\lambda x^1 + (1-\lambda) x^2\right) + b\right) =$$

$$f\left(\lambda\left(Ax^1 + b\right) + (1-\lambda)\left(Ax^2 + b\right)\right) \geq$$

$$\lambda f\left(Ax^1 + b\right) + (1-\lambda) f\left(Ax^2 + b\right) =$$

$$\lambda h\left(x^1\right) + (1-\lambda) h\left(x^2\right)$$

Hence, we establish that $h\left(\left(\lambda x^1 + (1-\lambda) x^2\right)\right) \geq \lambda h\left(x^1\right) + (1-\lambda) h\left(x^2\right)$, for any $x^1, x^2 \in R^m$ and $\lambda \in R$. Thus, $h: R^m \to R$ is also concave.