Economics 201B–Second Half
Lecture 10
Revised 4/24/09, Revisions Indicated by

The Index Theorem

- Throughout, we will denote a price in $\Delta = \{p \in \mathbf{R}^L_+ : \Sigma_{\ell=1}^L p_\ell = 1\}$ by p, and the associated price in $\mathbf{R}^{L-1}_{++}$ (with the assumption that the price of good L has been normalized to 1) by $\hat{p}$.

- *Definition*: If p^* is a regular equilibrium price, define

$$\text{index } (p^*) = (-1)^{L-1} \text{sign } \det D\hat{z}|_{\hat{p}^*}$$

- For $L = 2$,

$$\begin{aligned} \text{index } (p^*) &= (-1)^1 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \hat{z}'(\hat{p}^*) \end{aligned}$$

1. index $(p^*) = +1$ means that $\hat{z}'(\hat{p}^*) < 0$; that means demand is downward sloping, so we are in the "normal" case in which an increase in $\hat{z}$, the excess demand for good 1, results in an increase in the equilibrium price of good 1.

2. index $(p^*) = -1$ means that $\hat{z}'(\hat{p}^*) > 0$; that means demand is upward sloping, so we are in

the "abnormal" case in which an increase in $\hat{z}$, the excess demand for good 1, results in an decrease in the equilibrium price of good 1.

- For $L = 3$,

$$\begin{aligned} \text{index } (p^*) &= (-1)^2 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= \text{sign } \det D\hat{z}|_{\hat{p}^*} \end{aligned}$$

The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

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$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a counterclockwise rotation.

$$D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a rotation; counterclockwise (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \text{ index } (\hat{p}^*) = +1$$

and clockwise (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index } (\hat{p}^*) = -1$$

- For $L = 4 * *$,

$$\begin{aligned} \text{index } (p^*) &= (-1)^3 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \det D\hat{z}|_{\hat{p}^*} \end{aligned}$$

The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

–

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

is a right-handed system.

$$\det D\hat{z}|_{\hat{p}^*}\begin{pmatrix}1\\0\\0\end{pmatrix}, \det D\hat{z}|_{\hat{p}^*}\begin{pmatrix}0\\1\\0\end{pmatrix}, \det D\hat{z}|_{\hat{p}^*}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

is right-handed (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \text{ index }(\hat{p}^*) = -1$$

and left-handed (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index }(\hat{p}^*) = +1$$

- *Connection to Tatonnement Stability*: Consider the Tatonnement Price Dynamics

$$\frac{d\hat{p}}{dt} = \hat{z}(\hat{p}) \tag{1}$$

– This is a nonlinear differential equation, but we can approximate its behavior near an equilibrium price $\hat{p}^*$ by considering the linear differential equation

$$\frac{d\hat{p}}{dt} = D\hat{z}|_{\hat{p}^*}(\hat{p} - \hat{p}^*) \tag{2}$$

– Let $\lambda_1, \ldots, \lambda_{L-1}$ be the eigenvalues of $D\hat{z}|_{\hat{p}^*}$.

– *Fact*:

$$\det D\hat{z}|_{\hat{p}^*} = \prod_{\ell=1}^{L-1} \lambda_\ell$$

This is obvious if the matrix is diagonalizable, but is true in general.

* Some of the eigenvalues are real; the others come in conjugate pairs.
* If $a + bi$ and $a - bi$ are a conjugate pair of eigenvalues

$$(a + bi)(a - bi) = a^2 + b^2 > 0$$

* Thus,

$$\text{sign } \det D\hat{z}|_{\hat{p}^*} = \prod_{\lambda_\ell \in \mathbf{R}} \text{sign } (\lambda_\ell)$$

is the product of the signs of the real eigenvalues.

* Each complex eigenvalue represents a rotation, which does not change orientation.
* Each real, negative eigenvalue represents a change of orientation. Orientation is unchanged if there are an even number of real, negative eigenvalues.

– Equation (1) is locally asymptotically stable near $\hat{p}^*$ if * all solutions to Equation (2) converge to

$\hat{p}^*$, **which is true if and only if

$$\Re(\lambda_1) < 0, \ldots, \Re(\lambda_{L-1}) < 0$$

**If $\Re(\lambda_\ell) > 0$ for any ℓ, then Equation (1) is not locally asymptotically stable.

**Suppose $L-1$ is odd. Since there are an even number of complex eigenvalues, there are an odd number of real eigenvalues, so if all of them are negative, the determinant is negative and

$$\text{index } (\hat{p}^*) = (-1)^{L-1}\text{sign } \det D\hat{z}|_{\hat{p}^*} = +1$$

**On the other hand, suppose $L-1$ is even. Since there are an even number of complex eigenvalues, there are an even number of real eigenvalues, so if all of them are negative, the determinant is positive and

$$\text{index } (\hat{p}^*) = (-1)^{L-1}\text{sign } \det D\hat{z}|_{\hat{p}^*} = +1$$

Thus, we have

$$\text{Tatonnement Stability near } \hat{p}^* \Rightarrow \text{index } (\hat{p}^*) = +1$$

but the converse is false. **Thus, the Index Theorem lets us quickly determine that some equilibria are unstable, and allow us to concentrate a search for stable equilibria on those with index $+1$, which *might* be stable.

Theorem 1 (Index Theorem) *For any regular economy,*

$$\sum_{\hat{p}^* \in \mathbf{R}^{L-1}_{++}, \hat{z}(\hat{p}^*)=0} \text{index}\,(\hat{p}^*) = +1$$

Corollary 2 *For any regular economy, there are an odd number of equilibria. Since* 0 *is even, every regular economy has an equilibrium.*

Intuition behind Index Theorem: index $(\hat{p}^*)$ indicates the direction in which $\hat{z}$ passes through zero near $\hat{p}^*$. The Boundary Condition implies that $\hat{z}$ starts on one side of zero and ends up on the other side of zero, so every equilibrium price with index -1 must be paired with an equilibrium price with index +1, and exactly one equilibrium price with index +1 must be left unpaired.

Genericity: Almost All Economies are Regular

Review notion of Lebesgue measure zero from 204: This is a natural formulation of the notion that A is a small set:

"If you choose $x \in \mathbf{R}^n$ at random,
the probability that $x \in A$ is zero."

Regular and Critical Points and Values:
Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \to \mathbf{R}^m$ is differentiable at $x \in X$. Then $df_x \in L(\mathbf{R}^n, \mathbf{R}^m)$, so

$$\text{rank }(df_x) \leq \min\{m, n\}$$

- x is a *regular point* of f if $\text{rank }(df_x) = \min\{m, n\}$.
- x is a *critical point* of f if $\text{rank }(df_x) < \min\{m, n\}$.
- y is a *critical value* of f if there exists $x \in X$, $f(x) = y$, x is a critical point of y.
- y is a *regular value* of f if y is not a critical value of f (notice this has the counterintuitive implication that if $y \notin f(X)$, then y is automatically a regular value of f).

A function may have many critical points; for example, if a function is constant on an interval, then every element of the interval is a critical point. But it can't have many critical *values*.

Theorem 3 (2.4, Sard's Theorem) *Let $X \subseteq \mathbf{R}^n$ be open, $f : X \to \mathbf{R}^m$, f is C^r with $r \geq 1 +$* $\max\{0, n-m\}$. *Then the set of all critical values of f has Lebesgue measure zero.*

Recall that our definition of critical point differed from de la Fuente's in the case $m > n$. If $m > n$, then every $x \in X$ is critical using de la Fuente's definition, because

$$\text{rank } Df(x) \leq n < m$$

Consequently, every $y \in f(X)$ is a critical value, using de la Fuente's definition. This does not contradict Sard's Theorem, since one can show that $f(X)$ is a set of Lebesgue measure zero when $m > n$ and $f \in C^1$. The Transversality Theorem is a particularly convenient formulation of Sard's Theorem for our purposes:

Theorem 4 (2.5', Transversality Theorem) *Let*

$$\begin{aligned} X \times \Omega \ &\subseteq \ \mathbf{R}^{n+p} \textit{be open} \\ F : X \times \Omega &\rightarrow \mathbf{R}^m \in C^r \\ &\qquad \textit{with } r \geq 1 + \max\{0, n-m\} \end{aligned}$$

If

$$F(x,\omega) = 0 \Rightarrow DF(x,\omega) \textit{ has rank } m$$

then for all ω *except for a set of Lebesgue measure zero,*

$$F(x,\omega) = 0 \Rightarrow D_x F(x,\omega) \textit{has rank } m$$

In particular, if $m = n$*, there is a local implicit function*

$$x^*(\omega)$$

characterized by

$$F(x^*(\omega), \omega) = 0$$

x^* is C^r and the equilibrium correspondence is lower hemicontinuous.

Remark: If $n < m$, rank $D_x F(x, \omega) \leq \min\{m, n\} = n < m$. Therefore,

$$\begin{aligned} & (F(x,\omega) = 0 \Rightarrow DF(x,\omega) \text{ has rank m}) \\ \Rightarrow \ & \text{for all } \omega \text{ except for a set of Lebesgue measure zero} \\ & F(x,\omega) = 0 \text{ has no solution} \end{aligned}$$