

Lecture 10

The Index Theorem

- *Definition:* If p^* is a regular equilibrium price, define

$$\text{index}(p^*) = (-1)^{L-1} \text{sign det } D\hat{z}|_{\hat{p}^*}$$

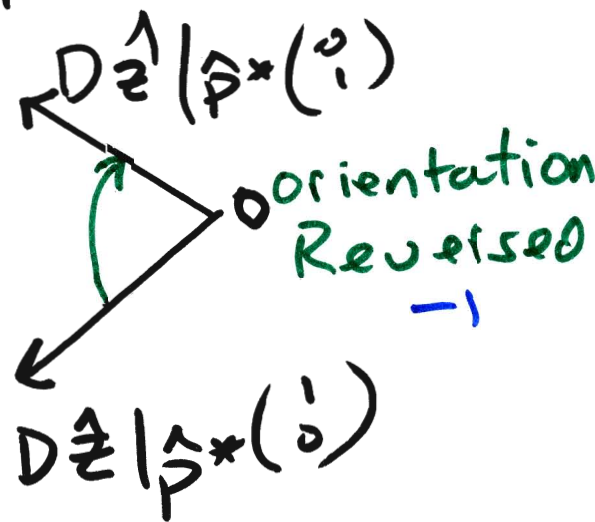
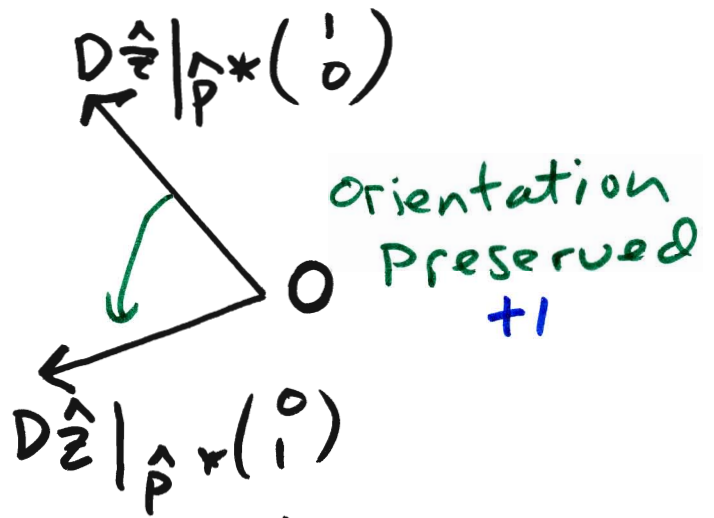
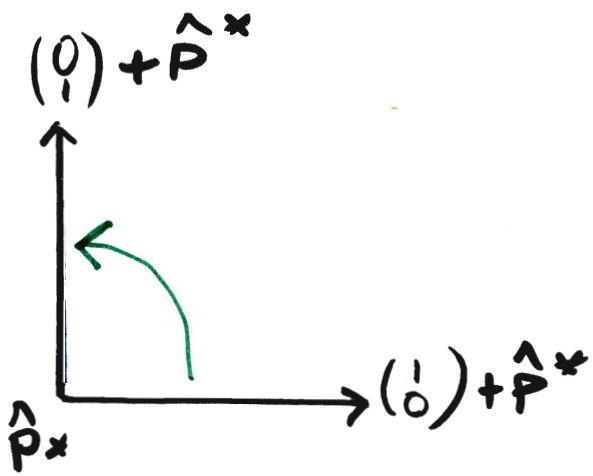
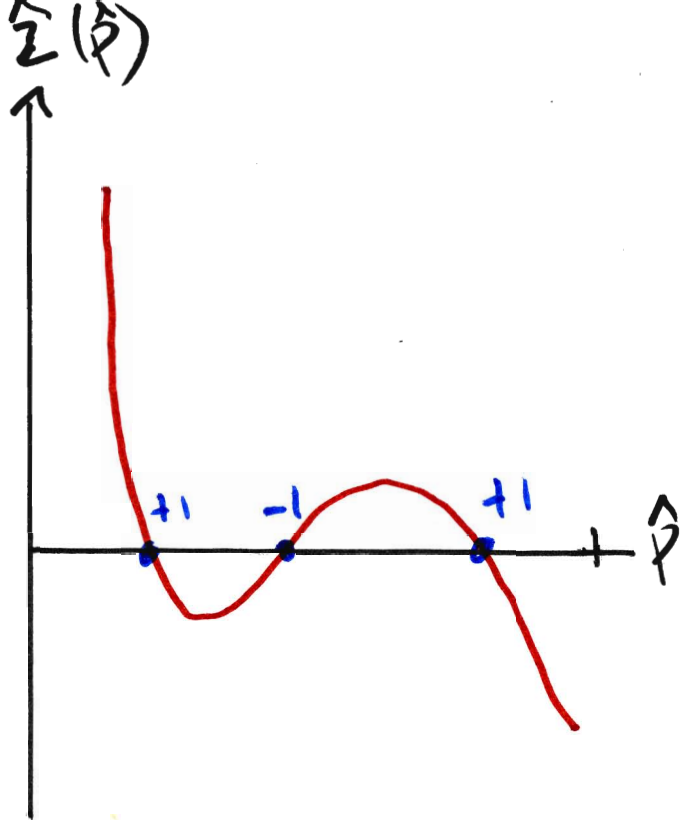
- For $L = 2$,

$$\begin{aligned} \text{index}(p^*) &= (-1)^1 \text{sign det } D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \hat{z}'(\hat{p}^*) \end{aligned}$$

1. $\text{index}(p^*) = +1$ means that $\hat{z}'(\hat{p}^*) < 0$; that means demand is downward sloping, so we are in the “normal” case in which an increase in \hat{z} , the excess demand for good 1, results in an increase in the equilibrium price of good 1.
2. $\text{index}(p^*) = -1$ means that $\hat{z}'(\hat{p}^*) > 0$; that means demand is upward sloping, so we are in the “abnormal” case in which an increase in \hat{z} , the excess demand for good 1, results in a decrease in the equilibrium price of good 1.

- For $L = 3$,

$$\begin{aligned} \text{index}(p^*) &= (-1)^2 \text{sign det } D\hat{z}|_{\hat{p}^*} \\ &= \text{sign det } D\hat{z}|_{\hat{p}^*} \end{aligned}$$



The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

—

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a counterclockwise rotation.

$$D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is obtained from } D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by a rotation; counterclockwise (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \text{ index } (\hat{p}^*) = +1$$

and clockwise (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index } (\hat{p}^*) = -1$$

- For $L = 3$,

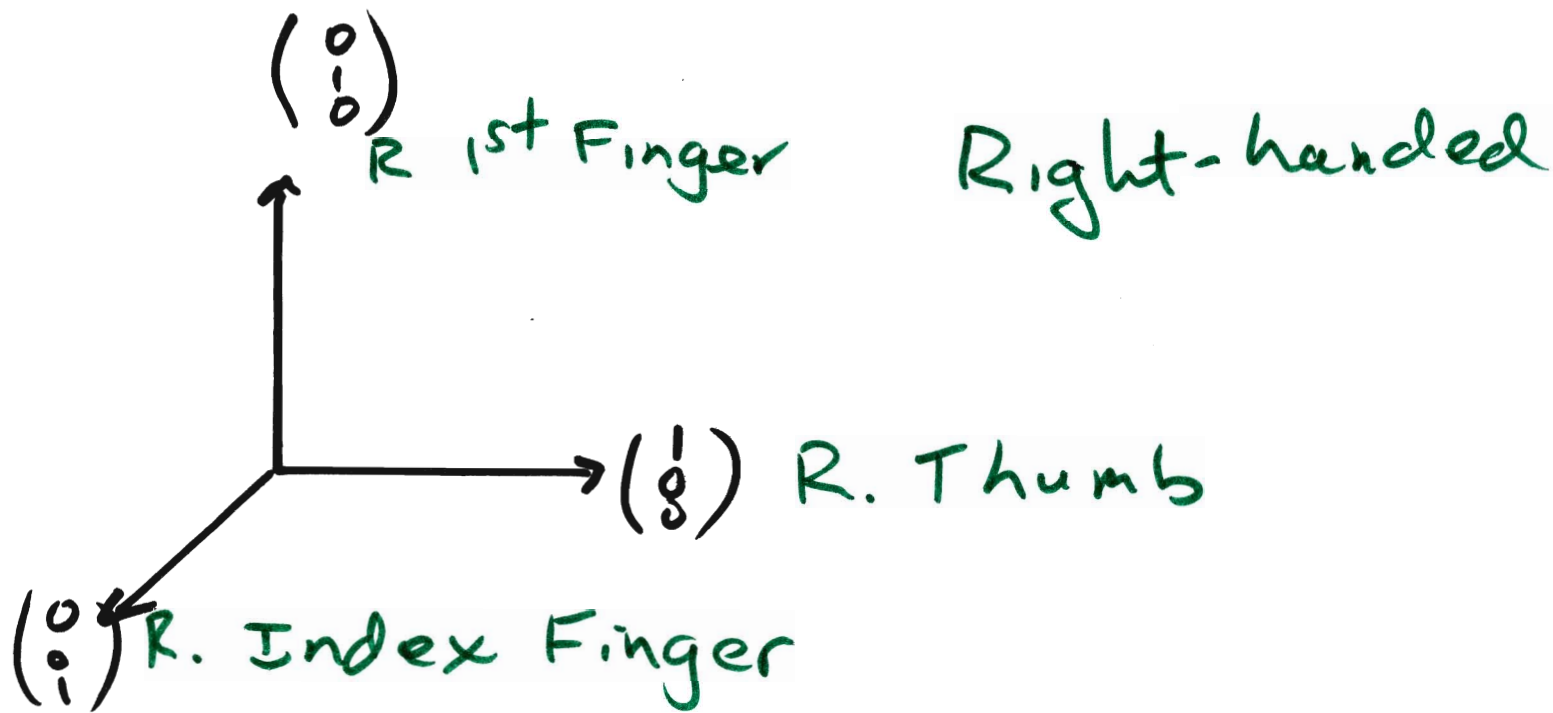
$$\begin{aligned} \text{index } (p^*) &= (-1)^3 \text{sign } \det D\hat{z}|_{\hat{p}^*} \\ &= -\text{sign } \det D\hat{z}|_{\hat{p}^*} \end{aligned}$$

The sign of the determinant is +1 if orientation is preserved, -1 if orientation is reversed:

—

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is a right-handed system.



L First Finger

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Left-handed

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ L Index Finger

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ L. Thumb

$$\det D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \det D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \det D\hat{z}|_{\hat{p}^*} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is right-handed (orientation preserved) if

$$\det D\hat{z}|_{\hat{p}^*} > 0, \text{ index } (\hat{p}^*) = -1$$

and left-handed (orientation reversed) if

$$\det D\hat{z}|_{\hat{p}^*} < 0, \text{ index } (\hat{p}^*) = +1$$

- *Connection to Tatonnement Stability:* Consider the Tatonnement Price Dynamics

$$\frac{d\hat{p}}{dt} = \hat{z}(\hat{p}) \tag{1}$$

- This is a nonlinear differential equation, but we can approximate its behavior near an equilibrium price \hat{p}^* by considering the linear differential equation

$$\frac{d\hat{p}}{dt} = D\hat{z}|_{\hat{p}^*} (\hat{p} - \hat{p}^*) \tag{2}$$

- Let $\lambda_1, \dots, \lambda_{L-1}$ be the eigenvalues of $D\hat{z}|_{\hat{p}^*}$.

- *Fact:*

$$\det D\hat{z}|_{\hat{p}^*} = \prod_{\ell=1}^{L-1} \lambda_{\ell}$$

This is obvious if the matrix is diagonalizable, but is true in general.

* Some of the eigenvalues are real; the others come in conjugate pairs.

* If $a + bi$ and $a - bi$ are a conjugate pair of eigenvalues

$$(a + bi)(a - bi) = a^2 + b^2 > 0$$

* Thus,

$$\text{sign det } D\hat{z}|_{\hat{p}^*} = \prod_{\lambda_\ell \in \mathbf{R}} \text{sign}(\lambda_\ell)$$

is the product of the signs of the real eigenvalues.

* Each complex eigenvalue represents a rotation, which does not change orientation.

* Each real, negative eigenvalue represents a change of orientation. Orientation is unchanged if there are an even number of real, negative eigenvalues.

– Equation (1) is locally asymptotically stable near \hat{p}^* if and only if all solutions to Equation (2) converge to \hat{p}^* if and only if

$$\Re(\lambda_1) < 0, \dots, \Re(\lambda_{L-1}) < 0$$

Since there are an even number of complex eigenvalues, this last condition implies that

$$\text{sign det } D\hat{z}|_{\hat{p}^*} = (-1)^{L-1} \Rightarrow \text{index}(\hat{p}^*) = +1$$

Thus, we have

$$\text{Tatonnement Stability near } \hat{p}^* \Rightarrow \text{index}(\hat{p}^*) = +1$$

but the converse is false.

Theorem 1 (Index Theorem) *For any regular economy,*

$$\sum_{\hat{p}^* \in \mathbf{R}_{++}^{L-1}, \hat{z}(\hat{p}^*)=0} \text{index}(\hat{p}^*) = +1$$

Corollary 2 *For any regular economy, there are an odd number of equilibria. Since 0 is even, every regular economy has an equilibrium.*

Intuition behind Index Theorem: $\text{index}(\hat{p}^*)$ indicates the direction in which \hat{z} passes through zero near \hat{p}^* .

The Boundary Condition implies that \hat{z} starts on one side of zero and ends up on the other side of zero, so every equilibrium price with index -1 must be paired with an equilibrium price with index +1, and exactly one equilibrium price with index +1 must be left unpaired.

Genericity: Almost All Economies are Regular

Review notion of Lebesgue measure zero from 204: This is a natural formulation of the notion that A is a small set:

“If you choose $x \in \mathbf{R}^n$ at random,
the probability that $x \in A$ is zero.”

Regular and Critical Points and Values:

Suppose $X \subseteq \mathbf{R}^n$ is open. Suppose $f : X \rightarrow \mathbf{R}^m$ is differentiable at $x \in X$. Then $df_x \in L(\mathbf{R}^n, \mathbf{R}^m)$, so

$$\text{rank}(df_x) \leq \min\{m, n\}$$

- x is a *regular point* of f if $\text{rank}(df_x) = \min\{m, n\}$.
- x is a *critical point* of f if $\text{rank}(df_x) < \min\{m, n\}$.
- y is a *critical value* of f if there exists $x \in X$, $f(x) = y$, x is a critical point of f .
- y is a *regular value* of f if y is not a critical value of f (notice this has the counterintuitive implication that if $y \notin f(X)$, then y is automatically a regular value of f).

A function may have many critical points; for example, if a function is constant on an interval, then every element of the interval is a critical point. But it can't have many critical values.

Theorem 3 (2.4, Sard's Theorem) *Let $X \subseteq \mathbf{R}^n$ be open, $f : X \rightarrow \mathbf{R}^m$, f is C^r with $r \geq 1 + \max\{0, n - m\}$. Then the set of all critical values of f has Lebesgue measure zero.*

Recall that our definition of critical point differed from de la Fuente's in the case $m > n$. If $m > n$, then every $x \in X$ is critical using de la Fuente's definition, because

$$\text{rank } Df(x) \leq n < m$$

Consequently, every $y \in f(X)$ is a critical value, using de la Fuente's definition. This does not contradict Sard's Theorem, since one can show that $f(X)$ is a set of Lebesgue measure zero when $m > n$ and $f \in C^1$.

The Transversality Theorem is a particularly convenient formulation of Sard's Theorem for our purposes:

Theorem 4 (2.5', Transversality Theorem) *Let*

$$X \times \Omega \subseteq \mathbf{R}^{n+p} \text{ be open}$$

$$F : X \times \Omega \rightarrow \mathbf{R}^m \in C^r$$

$$\text{with } r \geq 1 + \max\{0, n - m\}$$

If

$$F(x, \omega) = 0 \Rightarrow DF(x, \omega) \text{ has rank } m$$

then for all ω except for a set of Lebesgue measure zero,

$$F(x, \omega) = 0 \Rightarrow D_x F(x, \omega) \text{ has rank } m$$

In particular, if $m = n$, there is a local implicit function

$$x^*(\omega)$$

characterized by

$$F(x^*(\omega), \omega) = 0$$

and $x^ \in C^r$.*

Remark: If $n < m$, $\text{rank } D_x F(x, \omega) \leq \min\{m, n\} = n < m$. Therefore,

$$(F(x, \omega) = 0 \Rightarrow DF(x, \omega) \text{ has rank } m)$$

\Rightarrow for all ω except for a set of Lebesgue measure zero

$$F(x, \omega) = 0 \text{ has no solution}$$