Economics 201B–Second Half

Lecture 10, 4/15/10

Debreu's Theorem on Determinacy of Equilibrium

Definition 1 Let $F : \mathbf{R}_+^{L-1} \times \mathbf{R}_+^{LI} \to \mathbf{R}^{L-1}$ be defined by

$$F(\hat{p}, \omega) = \hat{z}(\hat{p}) \text{ when the endowment is } \omega$$

The *Equilibrium Price Correspondence* $E : \mathbf{R}_+^{LI} \to \mathbf{R}_{++}^{L-1}$ is defined by

$$E(\omega) = \left\{ \hat{p} \in \mathbf{R}_{++}^{L-1} : F(\hat{p}, \omega) = 0 \right\}$$

Proposition 2 *The Equilibrium Price Correspondence has closed graph.*

Proof: A version of this is on Problem Set 5. ■

Remark 3 If $\omega_n \to \omega$, it follows that the aggregate endowment $\bar{\omega}_n \to \bar{\omega}$. If $\bar{\omega} \in \mathbf{R}_{++}^L$, then an elaboration of the proof of the boundary condition on excess demand shows that $\bigcup_{n \in \mathbf{N}} E(\omega_n)$ is contained in a compact subset of $\mathbf{R}_{++}^{L-1}$, so in fact E is upperhemicontinuous at every ω such that $\bar{\omega} \in \mathbf{R}_{++}^L$.

Corollary 4 (Debreu) *Fix $\succeq_1, \ldots, \succeq_I$ so that*

$$D_i(p, \omega) \textit{ is a } C^1 \textit{ function of } p, \omega_i$$

and aggregate excess demand satisfies the hypotheses of the Debreu-Gale-Kuhn-Nikaido Lemma. Then there is a closed set $\Omega' \subset \mathbf{R}_+^{LI}$ of Lebesgue measure zero such that whenever $\omega_0 \in \mathbf{R}_+^{LI} \setminus \Omega'$,

- *the economy with preferences $\succeq_1, \ldots, \succeq_I$ and endowment ω is regular, so $E(\omega)$ is finite and odd;*

- *if* $E(\omega_0) = \{\hat{p}_1^*, \ldots, \hat{p}_N^*\}$*, then there is an open set* W *containing* ω_0 *and* C^1 *functions* $h_1, \ldots, h_N$ *such that, for all* $\omega \in W$*,*

  $$E(\omega) = \{h_1(\omega), \ldots, h_N(\omega)\}$$

  so E *is upper hemicontinuous and lower hemicontinuous at* ω*.*

Proof:

- *Claim*: For all $\omega \gg 0$ and price $\hat{p} \in \mathbf{R}_{++}^{L-1}$, and for each i,

  $$\text{rank } D_{\omega_i} F(\hat{p}, \omega) \geq L - 1$$

  Why? Let

  $$p = (\hat{p}, 1)$$

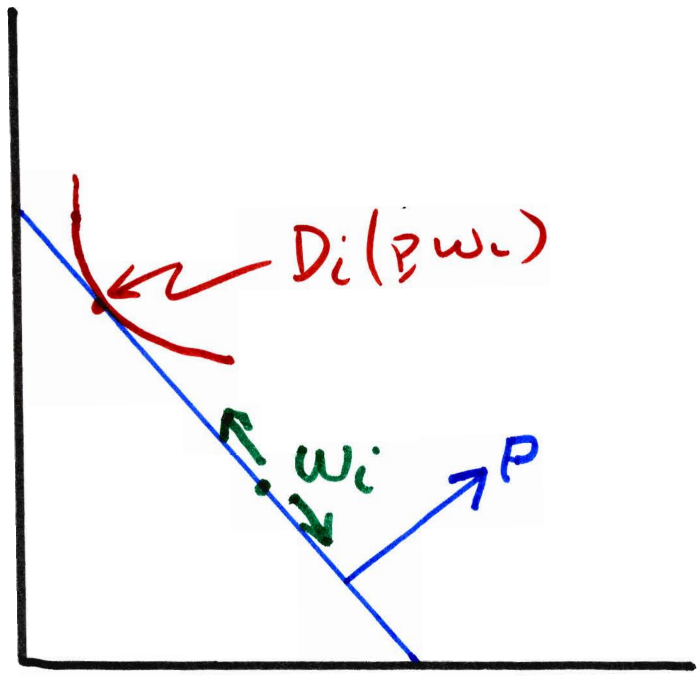
  Form an orthonormal basis $V = \{v_1, \ldots, v_L\}$ of $\mathbf{R}^L$ such that $v_1 = p/|p|$; thus, $\{v_2, \ldots, v_L\}$ will be an orthonormal basis of the hyperplane

  $$H = \{x \in \mathbf{R}^L : p \cdot x = 0\}$$

  of all vectors perpendicular to p. Let E_i denote the excess demand of agent i.

  $$E_i(p, \omega) = \sum_{\ell=1}^{L} (E_i(p, \omega) \cdot v_i) \, v_i = \sum_{\ell=2}^{L} (E_i(p, \omega) \cdot v_i) \, v_i$$

  since $E_i(p, \omega) \cdot p = 0$ by Walras' Law. Changing ω_i to $\omega_i + v_\ell$ ($\ell = 2, \ldots, L$) leaves the budget set unchanged, and hence leaves $D_i(p, \omega_i)$ unchanged, hence changes $E_i(p, \omega)$ by $-v_\ell$.



Di(p, wi)
wi
p

Then using the basis V for the domain and range,

$$D_{\omega_i}E_i(p,\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ ? & -1 & 0 & 0 & \cdots & 0 & 0 \\ ? & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ ? & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

The terms in first column other than the first entry come from the income effects in the Slutsky decomposition; we don't need to determine them. Obviously,

$$\text{rank } D_{\omega_i}E_i(p,\omega) = L-1$$

- The rank of the Jacobian matrix is independent of the basis, so when computed with respect to the standard basis,

$$\text{rank } D_{\omega_i}E_i(p,\omega) = L-1$$

But in the standard basis, $D_{\omega_i}F(\hat{p},\omega)$ consists of the first $L-1$ rows of $D_{\omega_i}E_i(p,\omega)$. By Walras' Law, the last row of the matrix is a linear combination of the first $L-1$ rows, so

$$\text{rank } D_{\omega_i}F(\hat{p},\omega) = L-1$$

- Since the range of F is $\mathbf{R}^{L-1}$,

$$\begin{aligned} L-1 &\geq \text{rank } DF(\hat{p},\omega) \\ &\geq \text{rank } D_{\omega_i}F(\hat{p},\omega) \\ &\geq L-1 \end{aligned}$$

so

$$\text{rank } DF(\hat{p},\omega) = L-1$$

- Let

$$\Omega'' = \left\{ \omega \in \mathbf{R}^{LI}_{++} : \exists_{\hat{p} \in \mathbf{R}^{L-1}_{++}} F(\hat{p}, \omega) = 0, \det D_{\hat{p}} \hat{z}(\hat{p}, \omega) = 0 \right\}$$

denote the set of endowments for which the resulting economy is not regular. By the Transversality Theorem, Ω'' has Lebesgue measure zero. Suppose we are given a sequence $\omega_n \in \Omega''$ with $\omega_n \to \omega \in \mathbf{R}^{LI}_{++}$. Choose $\hat{p}_n \in E(\omega_n)$ such that $\det D_{\hat{p}_n} \hat{z}(\hat{p}_n, \omega_n) = 0$. By Remark 3, there is a compact subset $\hat{K}$ of $\mathbf{R}^{L-1}_{++}$ such that $\cup_{n \in \mathbf{N}} E(\omega_n) \subset \hat{K}$. Thus, we can find a subsequence $\hat{p}_{n_k}$ converging to $\hat{p} \in \mathbf{R}^{LI}_{++}$.

$$\begin{aligned} \det D_{\hat{p}} \hat{z}(\hat{p}, \omega) &= \lim_{k \to \infty} \det D_{\hat{p}_{n_k}} \hat{z}(\hat{p}_{n_k}, \omega_{n_k}) \\ &= 0 \end{aligned}$$

so Ω'' is relatively closed in $\mathbf{R}^{L-1}_{++}$.

- Let

$$\Omega' = \Omega'' \cup \left(\mathbf{R}^{LI}_{+} \setminus \mathbf{R}^{LI}_{++} \right)$$

$\mathbf{R}^{LI}_{+} \setminus \mathbf{R}^{LI}_{++}$ is a set of Lebesgue measure zero, so Ω' is set of Lebesgue measure zero. Clearly Ω' is closed.

- If $\omega_0 \notin \Omega'$, the economy is regular, so $E(\omega_0)$ is finite and odd.

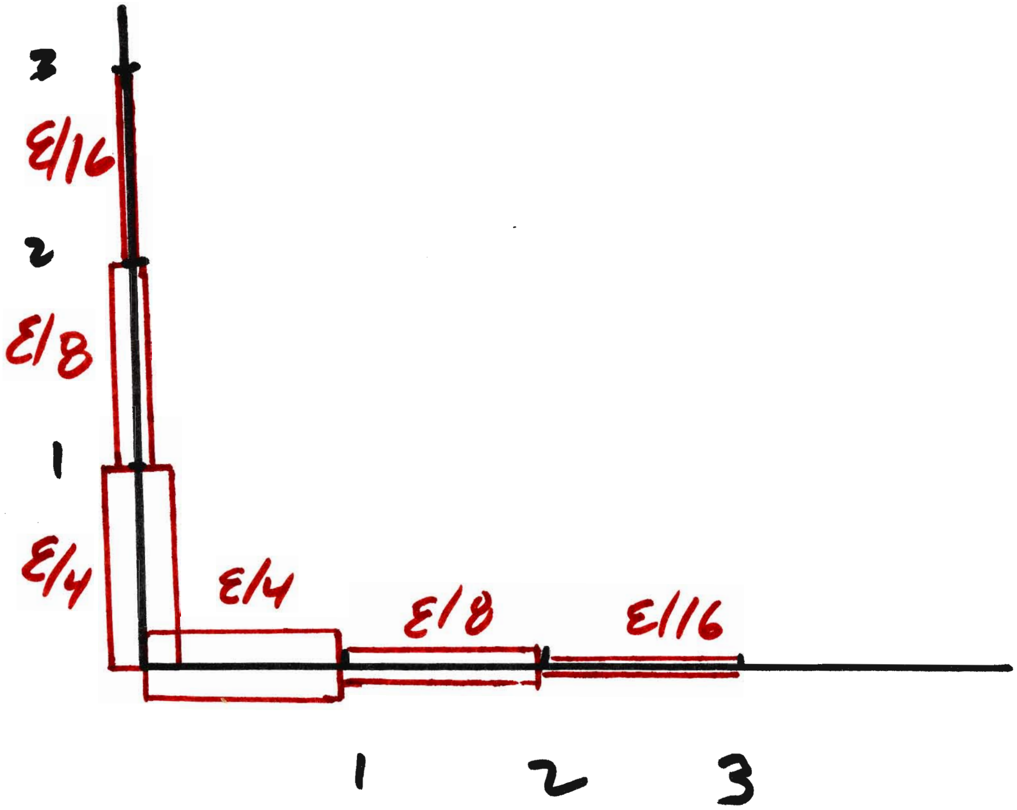
  - Let

  $$E(\omega_0) = \{\hat{p}^*_1, \ldots, \hat{p}^*_N\}$$

  By the Implicit Function Theorem, there are open sets V_n, W_n with $\hat{p}^*_n \in V_n$ and $\omega_0 \in W_n$ and C^1 functions $h_n : W_n \to \mathbf{R}^{L-1}_{++}$ such that for $\omega \in W_n$,

  $$E(\omega) \cap V_n = \{h_n(\omega)\}$$

  - E is lower hemicontinuous at ω by the Transversality Theorem as we stated it. This also follows directly from the implicit functions in the previous bullet.


3
ε/16
2
ε/8
1
ε/4
ε/4
ε/8
ε/16
1
2
3

- Let

$$W_0 = W_1 \cap \cdots \cap W_N, \; V = V_1 \cup \cdots \cup V_N$$

  W_0 is open and $\omega_0 \in W_0$. For $\omega \in W_0$,

$$E(\omega) \cap V = \{h_1(\omega), \ldots, h_N(\omega)\}$$

- By Remark 3, E is upper hemicontinuous at ω.

■

Limitations:

- The assumption that demand is C^1 is strong, but fixable (Cheng, Mas-Colell).
- Since the boundary of $\mathbf{R}_+^{LI}$ has Lebesgue measure zero, the formulation effectively assumes

$$\omega \in \mathbf{R}_{++}^{LI}$$

  - Terrible assumption, most agents are endowed with few goods.
  - Natural Conjecture: You can set certain endowments =0 and, as long as you have enough degrees of freedom in the nonzero endowments, Debreu's Theorem still holds. False: example due to Minehart.
  - Solution: Perturb preferences as well as endowments. Need genericity notion on infinite-dimensional spaces. Debreu's Theorem holds generically in a topological notion of genericity (Mas-Colell) and a measure-theoretic notion of genericity (Anderson & Zame).

- For Finance, commodity differentiation, choice under uncertainty, need version of theorem for infinite-dimensional commodity spaces. Shannon and Zame showed that close analogue to Debreu's Theorem holds. The consumption set often has empty interior in these infinite-dimensional settings, so differentiability is problematic; Shannon and Zame find that the functions defining the movement of the equilibrium prices are Lipschitz.

Quick Romp Through 17.E,F,H

- *17.E*

  Theorem 5 (Sonnenschein-Mantel-Debreu) *Let K be a compact subset of Δ^0. Given $f : K \to \mathbf{R}^L$ satisfying*

  - *continuity*
  - *Walras' Law with Equality ($p \cdot f(p) = 0$)*

  there is an exchange economy with L consumers whose excess demand function, restricted to K, equals f.

  Proof: Elementary, but far from transparent. Individual preferences may be made arbitrarily nice.

  ■

  Corollary 6 *There are no comparative statics results for Walrasian Equilibrium in the Arrow-Debreu model; more assumptions are needed.*

- *17.F, Uniqueness*:

  There are no results known under believeable assumptions on individual preferences.

- *17.H, Tatonnement Stability*:

$$\begin{aligned} \frac{d\hat{p}}{dt} &= \hat{z}(\hat{p}) \text{ on } \mathbf{R}^{L-1}_{++} \\ \frac{dp}{dt} &= E(p) \text{ on } \Delta^0_2 = \left\{p \in \mathbf{R}^L_{++} : \|p\|_2 = 1\right\} \end{aligned}$$

  We would like to know that the solutions converge to the equilibrium price. Scarf gave an example of a non-pathological exchange economy in which the solutions all circle around the unique Walrasian equilibrium price. There are no known stability results based on reasonable assumptions on individual preferences. Index $= +1$ is necessary but not sufficient for stability.

- *Modern Approach to Uniqueness and Stability*:

  Assumptions on the *Distribution* of Agents' Characteristics.

  Law of Demand:

$$(p - q) \cdot (z(p) - z(q)) \leq 0 \text{ with strict inequality if } p \neq q$$

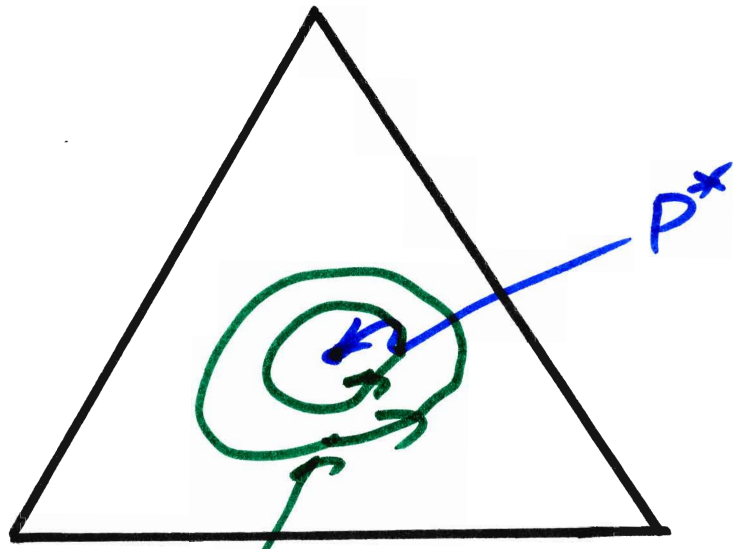
  The Law of Demand implies uniqueness of equilibrium and Tatonnement stabilty.

  - Hildenbrand:

    * If, for each preference, the density of the income distribution among people holding that preference is decreasing, then the Law of Demand holds.

    * Idea: If demand for a good is a decreasing function of income at some income level, it must first have been an increasing function at lower income levels. Decreasing density of income distribution implies that overall, the increasing part cancels out the decreasing part.

Scarf Example



Solutions to

$$\frac{dp}{dt} = E(p)$$

Hildenbrand

- Grandmont and Quah:
  - If preferences are dispersed, the Law of Demand holds.
  - Fix a preference $\succeq$. Given $\lambda \in \mathbf{R}^L_{++}$, define $\succeq_\lambda$ by

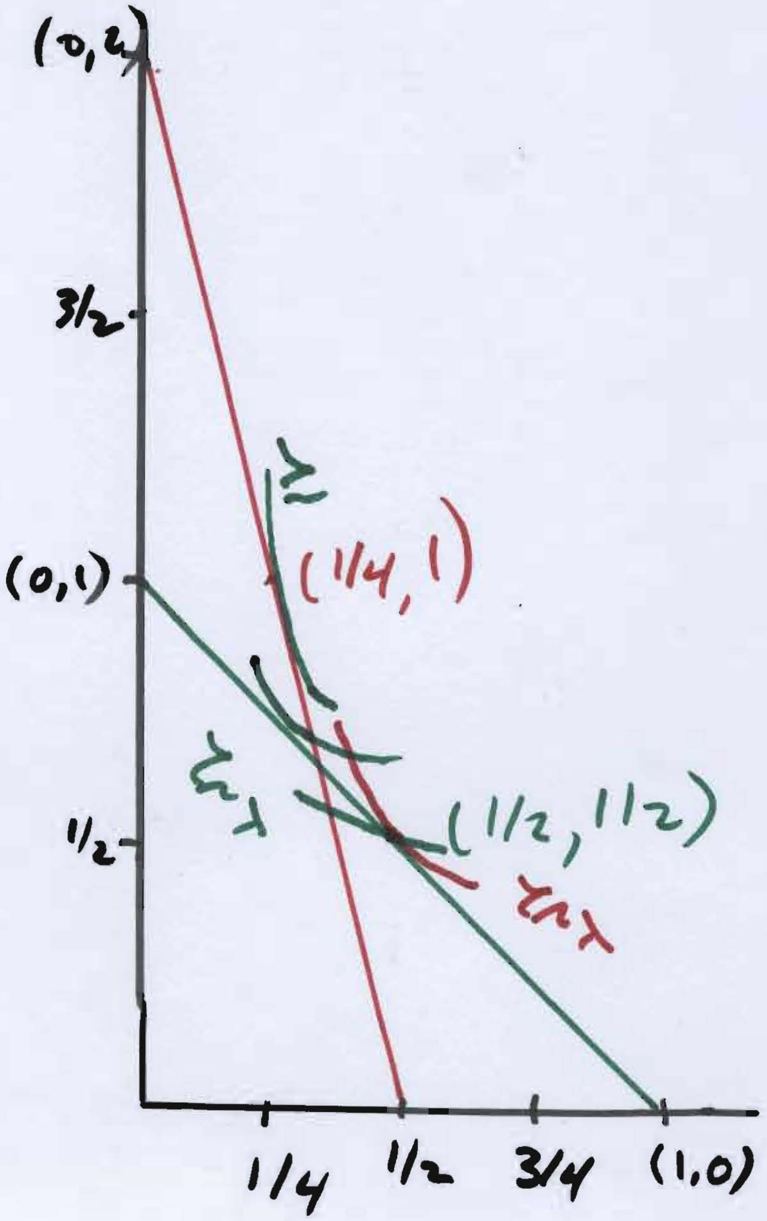
$$x \succeq_\lambda y \Leftrightarrow (\lambda_1 x_1, \lambda_2 x_2) \succeq (\lambda_1 y_1, \lambda_2 y_2)$$

    $\succeq_\lambda$ has the marginal rates of substitution shifted by the rescaling by λ. Let

$$\mathcal{P}(\succeq) = \left\{ \succeq_\lambda : \lambda \in \mathbf{R}^L_{++} \right\}$$

  - Grandmont:
    - Suppose that for every $\succeq$, among the people whose preferences lie in $\mathcal{P}(\succeq)$, the distribution of λ is sufficiently dispersed. Then the economy satisfies the Law of Demand.
    - Idea: For a given preference, demand may be upward sloping in price at certain prices, but given the Boundary Condition, it must be downward sloping at most prices. The prices at which demand is upward sloping are shifted by λ. If the distribution of λ is sufficiently dispersed, then for every p, most people will have downward sloping demand and they will outweigh the few that have upward sloping demand.
  - Quah:
    - Showed that a much weaker dispersion condition suffices to establish the Law of Demand.
    - Showed that in 1-good economies (Finance), reasonable conditions on how much each individual's coefficient of relative risk aversion varies over the relevant income range imply the Law of Demand.

Grandmont



$\lambda = (1/2, 2)$