Economics 201B–Second Half
Lecture 11
Revised 5/2/09, Revisions Indicated by and Sticky Notes

Tranversality Theorem, and Generic Regularity Recall the statement of the Transversality Theorem from 204, and the end of last lecture:

Theorem 1 (2.5', Transversality Theorem) *Let*

$$X \times \Omega \subseteq \mathbf{R}^{n+p} \textit{be open}$$
$$F : X \times \Omega \to \mathbf{R}^m \in C^r$$
$$\textit{with } r \geq 1 + \max\{0, n - m\}$$

If

$$F(x, \omega) = 0 \Rightarrow DF(x, \omega) \textit{ has rank } m$$

then for all ω except for a set of Lebesgue measure zero,

$$F(x, \omega) = 0 \Rightarrow D_x F(x, \omega) \textit{ has rank } m$$

In particular, if $m = n$, there is a local implicit function

$$x^*(\omega)$$

characterized by

$$F(x^*(\omega), \omega) = 0$$

x^* is a C^r function of ω, and the correspondence

$$\omega \to \{x : F(x, \omega) = 0\}$$

is lower hemicontinuous at ω.

Interpretation of Tranversality Theorem

- Ω: a set of parameters. In our case, $\Omega = \mathbf{R}^{LI}_{++}$, the set of strictly positive endowment profiles, $p = LI$.
- X: a set of variables. In our case, $X = \mathbf{R}^{L-1}_{++}$, the set of strictly positive prices normalized by $p_L = 1$.
- $\mathbf{R}^m$ is the range of F. In our case, $F(x, \omega) = \hat{z}(x)$, when the endowment profile is ω, $m = n = L - 1$.
- $F(x, \omega) = 0$ says that x is an equilibrium price when the endowment profile is ω.
- rank $DF(x, \omega) = m = L - 1$ says that, by adjusting either the prices x or the endowments ω, it is possible to move $F = \hat{z}$ in any direction in $\mathbf{R}^{L-1}$
- rank $D_x F(x, \omega) = m = L-1$ says det $D_x F(x, \omega) \neq 0$, which says the economy is regular and is the hypothesis of the Implicit Function Theorem. This will tell us that the equilibrium prices are given by a finite number of implicit functions of the parameters (endowments).

- Parameters of any given economy are fixed. However, we want to study the *set* of parameters for which the resulting economy is well-behaved.

- Theorem says the following:

  > "If, whenever $\hat{z}(\hat{p}^*) = 0$, it is possible by perturbing the endowments and adjusting the prices to move $\hat{z}$ in any direction in $\mathbf{R}^{L-1}$, then for almost all endowments, the resulting economy is regular, and hence there are finitely many equilibrium prices and the equilibrium prices are implicitly defined C^r functions of the endowments, and the equilibrium price correspondence is lower hemicontinuous."

- If $n < m$, rank $D_x F(x, \omega) \leq \min\{m, n\} = n < m$. Therefore,

$$
\begin{aligned}
& (F(x,\omega) = 0 \Rightarrow DF(x,\omega) \text{ has rank m}) \\
\Rightarrow \; & \text{for all } \omega \text{ except for a set of Lebesgue measure zero} \\
& F(x,\omega) = 0 \text{ has no solution}
\end{aligned}
$$

Definition 2 Let $F : \mathbf{R}_+^{L-1} \times \mathbf{R}_+^{LI} \to \mathbf{R}^{L-1}$ be defined by

$$F(\hat{p}, \omega) = \hat{z}(\hat{p}) \text{ when the endowment is } \omega$$

The *Equilibrium Price Correspondence* $E : \mathbf{R}^{LI}_{+} \to \mathbf{R}^{L-1}_{++}$ is defined by

$$E(\omega) = \left\{\hat{p} \in \mathbf{R}^{L-1}_{++} : F(\hat{p}, \omega) = 0\right\}$$

Proposition 3 *The Equilibrium Price Correspondence has closed graph.*

Proof: A version of this is on Problem Set 5. ∎

Remark 4 If $\omega_n \to \omega$, it follows that the aggregate endowment $\bar{\omega}_n \to \bar{\omega}$. If $\bar{\omega} \in \mathbf{R}^{L}_{++}$, then an elaboration of the proof of the boundary condition on excess demand shows that $\cup_{n\in\mathbf{N}} E(\omega_n)$ is contained in a compact subset of $\mathbf{R}^{L-1}_{++}$, so in fact E is upperhemicontinuous at every ω such that $\bar{\omega} \in \mathbf{R}^{L}_{++}$.

Corollary 5 (Debreu) *Fix $\succeq_1, \ldots, \succeq_I$ so that*

$$D_i(p, \omega) \text{ is a } C^1 \text{ function of } p, \omega_i$$

and aggregate excess demand satisfies the hypotheses of the Debreu-Gale-Kuhn-Nikaido Lemma. Then there is a closed set $\Omega' \subset \mathbf{R}^{LI}_{+}$ of Lebesgue measure zero such that whenever $\omega_0 \in \mathbf{R}^{LI}_{+} \setminus \Omega'$,

- *the economy with preferences $\succeq_1, \ldots, \succeq_I$ and endowment ω is regular, so $E(\omega)$ is finite and odd;*

- *if* $E(\omega_0) = \{\hat{p}_1^*, \ldots, \hat{p}_N^*\}$, *then there is an open set* W *containing* ω_0 *and* C^1 *functions* $h_1, \ldots, h_N$ *such that, for all* $\omega \in W$,

$$E(\omega) = \{h_1(\omega), \ldots, h_N(\omega)\}$$

so E *is upper hemicontinuous and lower hemicontinuous at* ω.

Proof:

- *Claim*: For all $\omega \gg 0$ and price $\hat{p} \in \mathbf{R}^{L-1}_{++}$, and for each i,

$$\text{rank } D_{\omega_i} F(\hat{p}, \omega) \geq L - 1$$

Why? Let

$$p = (\hat{p}, 1)$$

Suppose $\Delta\omega_i \in \mathbf{R}^L$ and

$$p \cdot \Delta\omega_i = 0$$

Changing

$$\omega_i \text{ to } \omega_i + \Delta\omega_i$$

leaves the budget set unchanged, and hence leaves $D_i(p, \omega_i)$ unchanged, hence changes $z(p, \omega)$ by $-\Delta\omega_i$. Thus, we have $L - 1$ degrees of freedom in moving F by perturbing endowments perpendicular to p, so

$$\text{rank } D_{\omega_i} F(\hat{p}, \omega) \geq L - 1$$

More formally, choose $v_1, \ldots, v_{L-1} \in \mathbf{R}^L$ such that $\{v_1, \ldots, v_{L-1}\}$ is a basis of $\left\{x \in \mathbf{R}^L : p \cdot x = 0\right\}$ and hence $\{p, v_1, \ldots, v_{L-1}\}$ is a basis for $\mathbf{R}^L$. * Let E_i denote the excess demand of agent i. Then with respect to this basis,

$$D_{\omega_i} E_i(p, \omega) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ ? & -1 & 0 & 0 & \cdots & 0 & 0 \\ ? & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ ? & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

* Since span $\{v_1, \ldots, v_{L-1}\}$ is perpendicular to p,

$$\begin{aligned} E_i(p, \omega_i) &= (p \cdot E_i(p, \omega_i))\, p + \sum_{\ell=1}^{L-1} \alpha_\ell v_\ell \\ &= (0)\, p + \sum_{\ell=1}^{L-1} \alpha_\ell v_\ell \end{aligned}$$

so the first row is zero by Walras' Law. The terms in the remainder of the first column come from the income effects in the Slutsky decomposition; we don't need to determine them. The $(L-1) \times (L-1)$ submatrix obtained by excluding the first row and first column arises from changes in ω_i that do not change income (or price), so the demand is constant and the change in excess demand is minus the change in endowment: this submatrix is the minus the $(L-1)\times(L-1)$ identity matrix.

Obviously,

$$\text{rank } D_{\omega_i}E_i(p,\omega) = L-1$$

- The rank of the Jacobian matrix is independent of the basis, so when computed with respect to the standard basis,

$$\text{rank } D_{\omega_i}E_i(p,\omega) = L-1$$

But in the standard basis, $D_{\omega_i}F(\hat{p},\omega)$ consists of the first $L-1$ rows of $D_{\omega_i}E_i(p,\omega)$. By Walras' Law, the last row of the matrix is a linear combination of the first $L-1$ rows, so

$$\text{rank } D_{\omega_i}F(\hat{p},\omega) = L-1$$

- Since the range of F is $\mathbf{R}^{L-1}$,

$$\begin{aligned} L-1 &\geq \text{rank } DF(\hat{p},\omega) \\ &\geq \text{rank } D_{\omega_i}F(\hat{p},\omega) \\ &\geq L-1 \end{aligned}$$

so

$$\text{rank } DF(\hat{p},\omega) = L-1$$

- Let

$$\Omega'' = \Big\{\omega \in \mathbf{R}^{LI}_{++} : \exists_{\hat{p}\in\mathbf{R}^{L-1}_{++}}\ F(\hat{p},\omega)=0, \det D_{\hat{p}}\hat{z}(\hat{p},\omega)$$

denote the set of endowments for which the resulting economy is not regular. By the Transversality Theorem, Ω'' has Lebesgue measure zero. Suppose we are given a sequence $\omega_n \in \Omega''$ with $\omega_n \to \omega \in \mathbf{R}^{LI}_{++}$. Choose $\hat{p}_n \in E(\omega_n)$ such that $\det D_{\hat{p}_n}\hat{z}(\hat{p}_n, \omega_n) = 0$. By Remark 4, there is a compact subset $\hat{K}$ of $\mathbf{R}^{L-1}_{++}$ such that $\cup_{n\in\mathbf{N}} E(\omega_n) \subset \hat{K}$. Thus, we can find a subsequence $\hat{p}_{n_k}$ converging to $\hat{p} \in \mathbf{R}^{LI}_{++}$.

$$\begin{aligned} \det D_{\hat{p}}\hat{z}(\hat{p}, \omega) &= \lim_{k\to\infty} \det D_{\hat{p}_{n_k}}\hat{z}(\hat{p}_{n_k}, \omega_{n_k}) \\ &= 0 \end{aligned}$$

so Ω'' is relatively closed in $\mathbf{R}^{L-1}_{++}$.

- Let

$$\Omega' = \Omega'' \cup \left(\mathbf{R}^{LI}_{+} \setminus \mathbf{R}^{LI}_{++}\right)$$

  $\mathbf{R}^{LI}_{+} \setminus \mathbf{R}^{LI}_{++}$ is a set of Lebesgue measure zero, so Ω' is set of Lebesgue measure zero. Clearly Ω' is closed.

- If $\omega_0 \notin \Omega'$, the economy is regular, so $E(\omega_0)$ is finite and odd. Let

$$E(\omega_0) = \{\hat{p}^*_1, \ldots, \hat{p}^*_N\}$$

  By the Implicit Function Theorem, there are open sets V_n, W_n with $\hat{p}^*_n \in V_n$ and $\omega_0 \in W_n$ and C^1 functions $h_n : W_n \to \mathbf{R}^{L-1}_{++}$ such that for $\omega \in W_n$,

$$E(\omega) \cap V_n = \{h_n(\omega)\}$$

This shows that E is lower hemicontinous at ω.

Let

$$W_0 = W_1 \cap \cdots \cap W_N, \; V = V_1 \cup \cdots \cup V_N$$

W_0 is open and $\omega_0 \in W_0$. For $\omega \in W_0$,

$$E(\omega) \cap V = \{h_1(\omega), \ldots, h_N(\omega)\}$$

By Remark 4, E is upper hemicontinuous at ω.

■

Limitations:

- The assumption that demand is C^1 is strong, but fixable (Cheng, Mas-Colell).
- Since the boundary of $\mathbf{R}_+^{LI}$ has Lebesgue measure zero, the formulation effectively assumes

$$\omega \in \mathbf{R}_{++}^{LI}$$

  – Terrible assumption, most agents are endowed with few goods.

  – Natural Conjecture: You can set certain endowments =0 and, as long as you have enough degrees of freedom in the nonzero endowments, Debreu's Theorem still holds. False: example due to Minehart.

- Solution: Perturb preferences as well as endowments. Need genericity notion on infinite-dimensional spaces. Debreu's Theorem holds generically in a topological notion of genericity (Mas-Colell) and a measure-theoretic notion of genericity (Anderson & Zame).

- For Finance, commodity differentiation, choice under uncertainty, need version of theorem for infinite-dimensional commodity spaces. Shannon and Zame showed that close analogue to Debreu's Theorem holds. The consumption set often has empty interior in these infinite-dimensional settings, so differentiability is problematic; Shannon and Zame find that the functions defining the movement of the equilibrium prices are Lipschitz.