Economics 201B–Second Half

Lecture 12

Quick Romp Through 17.E,F,H

- *17.E*

  Theorem 1 (Sonnenschein-Mantel-Debreu) *Let K be a compact subset of Δ^0. Given $f : K \to \mathbf{R}^L$ satisfying*

  - *continuity*
  - *Walras' Law with Equality ($p \cdot f(p) = 0$)*

  there is an exchange economy with L consumers whose excess demand function, restricted to K, equals f.

  Proof: Elementary, but far from transparent. Individual preferences may be made arbitrarily nice.

  ■

  Corollary 2 *There are no comparative statics results for Walrasian Equilibrium in the Arrow-Debreu model; more assumptions are needed.*

- *17.F, Uniqueness*:

  There are no results known under believeable assumptions on individual preferences.

- *17.H, Tatonnement Stability*:

$$\frac{d\hat{p}}{dt} = \hat{z}(\hat{p}) \text{ on } \mathbf{R}^{L-1}_{++}$$
$$\frac{dp}{dt} = E(p) \text{ on } \Delta^0_2 = \left\{p \in \mathbf{R}^L_{++} : \|p\|_2 = 1\right\}$$

  We would like to know that the solutions converge to the equilibrium price. Scarf gave an example of a non-pathological exchange economy in which the solutions all circle around the unique Walrasian equilibrium price. There are no known stability results based on reasonable assumptions on individual preferences. Index = +1 is necessary but not sufficient for stability.

- *Modern Approach to Uniqueness and Stability*:

  Assumptions on the *Distribution* of Agents' Characteristics.

  Law of Demand:

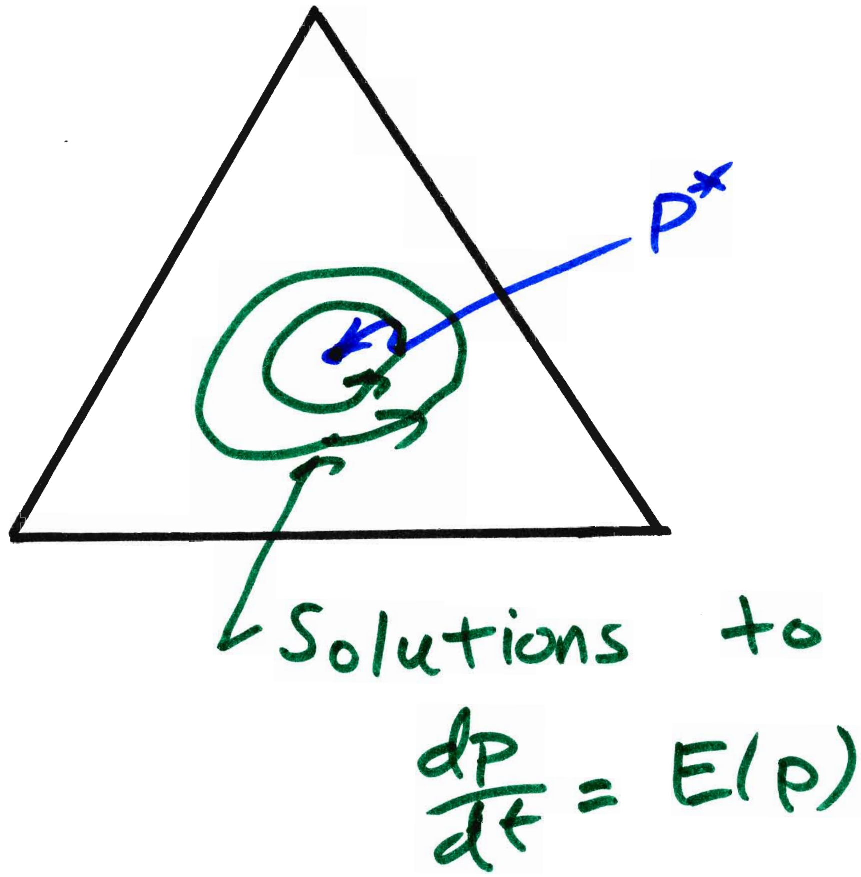
$$(p - q) \cdot (z(p) - z(q)) \leq 0 \text{ with strict inequality if } p \neq q$$

  The Law of Demand implies uniqueness of equilibrium and Tatonnement stabilty.

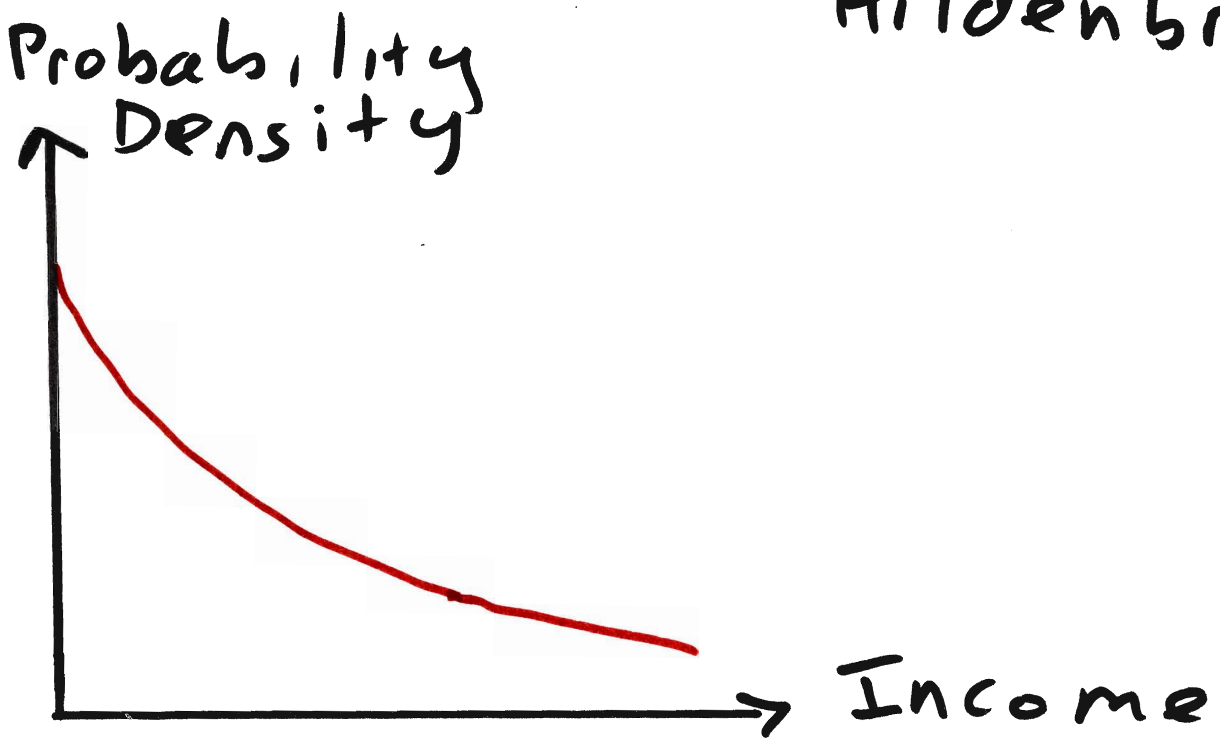
  - Hildenbrand:

    * If, for each preference, the density of the income distribution among people holding that preference is decreasing, then the Law of Demand holds.

Scarf Example



Hildenbrand



    * Idea: If demand for a good is a decreasing function of income at some income level, it must first have been an increasing function at lower income levels. Decreasing density of income distribution implies that overall, the increasing part cancels out the decreasing part.

- Grandmont and Quah:
    * If preferences are dispersed, the Law of Demand holds.
    * Fix a preference $\succeq$. Given $\lambda \in \mathbf{R}^L_{++}$, define $\succeq_\lambda$ by

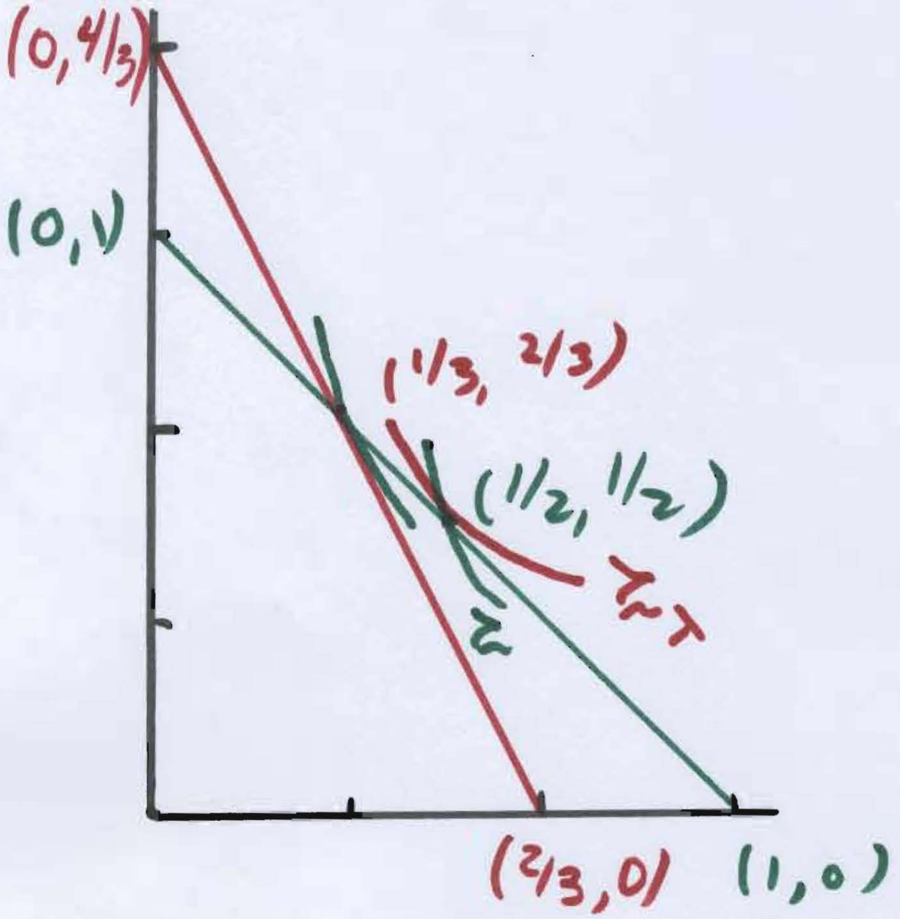
$$x \succeq_\lambda y \Leftrightarrow (\lambda_1 x_1, \lambda_2 x_2) \succeq (\lambda_1 y_1, \lambda_2 y_2)$$

      $\succeq_\lambda$ has the marginal rates of substitution shifted by the rescaling by λ. Let

$$\mathcal{P}(\succeq) = \left\{\succeq_\lambda : \lambda \in \mathbf{R}^L_{++}\right\}$$

    * Grandmont:
        · Suppose that for every $\succeq$, among the people whose preferences lie in $\mathcal{P}(\succeq)$, the distribution of λ is sufficiently dispersed. Then the economy satisfies the Law of Demand.

Grandmont



$\lambda = (2/3, 4/3)$

        - Idea: For a given preference, demand may be upward sloping in price at certain prices, but given the Boundary Condition, it must be downward sloping at most prices. The prices at which demand is upward sloping are shifted by λ. If the distribution of λ is sufficiently dispersed, then for every p, most people will have downward sloping demand and they will outweigh the few that have upward sloping demand.

    * Quah:

        - Showed that a much weaker dispersion condition suffices to establish the Law of Demand.

        - Showed that in 1-good economies (Finance), reasonable conditions on how much each individual's coefficient of relative risk aversion varies over the relevant income range imply the Law of Demand.

17.I and Handout: Nonconvex Preferences and Indivisibilities

- Why do we care?

- General story that justifies convexity (diminishing MRS along each indifference curve) works for "most" goods, but if there is a single pair of goods for which preferences are nonconvex, or if there is a single good which is indivisible, Existence of Walrasian Equilibrium and the Second Welfare Theorem fail.

  * $\frac{1}{2}$ house in Berkeley and $\frac{1}{2}$ house in SF
  * Two trips to Winnemucca are not preferable to one trip to Salt Lake if you like to ski.
  * Painting a room with orange and green stripes is not preferable to solid orange or solid green.

- **Theorem 3 (Shapley-Folkman)** *Suppose* $x \in \text{con}\,(A_1 + \cdots + A_I)$*, where* $A_i \subset \mathbf{R}^L$*. Then we may write* $x = a_1 + \cdots + a_I$*, where* $a_i \in \text{con}\,A_i$ *for all* i *and* $a_i \in A_i$ *for all but* L *values of* i*.*

  Proof: The proof is in the handout, it just uses the fact that $m \geq L + 1$ vectors in $\mathbf{R}^L$ must be linearly dependent.■

  Theorem 4 *Suppose we are given a pure exchange economy, where for each* $i = 1, \ldots, I$, $\succ_i$ *satisfies*

  1. *continuity:* $\{(x, y) \in \mathbf{R}^L_+ \times \mathbf{R}^L_+ : x \succ_i y\}$ *is relatively open in* $\mathbf{R}^L_+ \times \mathbf{R}^L_+$*;*
  2. *for each individual* i*, the consumption set is* $\mathbf{R}^L_+$*, i.e. each good is perfectly divisible, and each agent is capable of surviving on zero consumption;*

3. *acyclicity: there is no collection* $x_1, x_2, \ldots, x_m$ *such that* $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m \succ_i x_1$*;*

4. *strong monotonicity:*

$$x > y \Rightarrow x \succ_i y$$

Then there exists $p^* \gg 0$ *with* $0 \in \mathrm{con}\, E(p^*)$ *and* $x_i^* \in D_i(p^*)$ *such that*

$$\frac{1}{I}\sum_{\ell=1}^{L} p_\ell^* \left| \left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i \right)_\ell \right| \leq \frac{2L}{I} \max\{\|\omega_i\|_\infty : i = 1, \ldots, I\}$$

where $\|x\|_\infty = \max\{|x_1|, \ldots, |x_L|\}$.

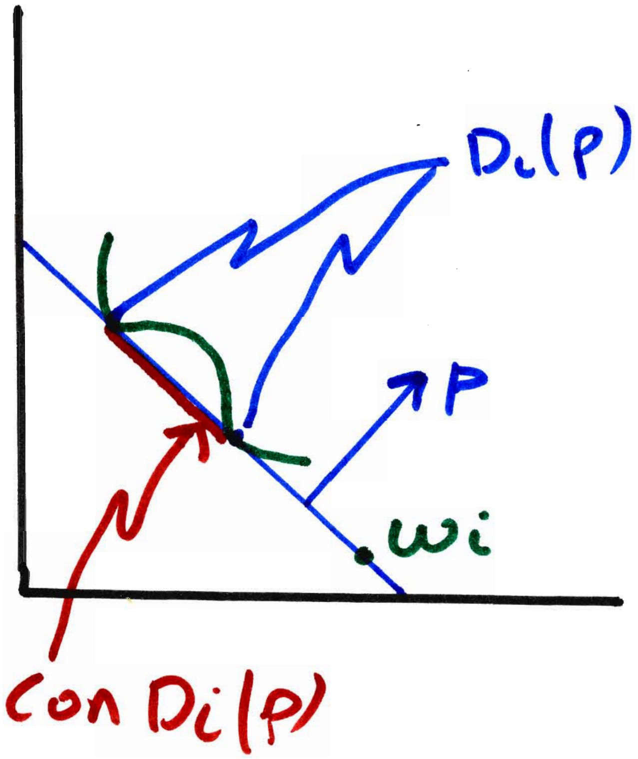
Theorem 5 *Suppose we are given a pure exchange economy, where for each* $i = 1, \ldots, I$, $\succ_i$ *satisfies*

1. *continuity:* $\{(x,y) \in \mathbf{R}_+^L \times \mathbf{R}_+^L : x \succ_i y\}$ *is relatively open in* $\mathbf{R}_+^L \times \mathbf{R}_+^L$*;*

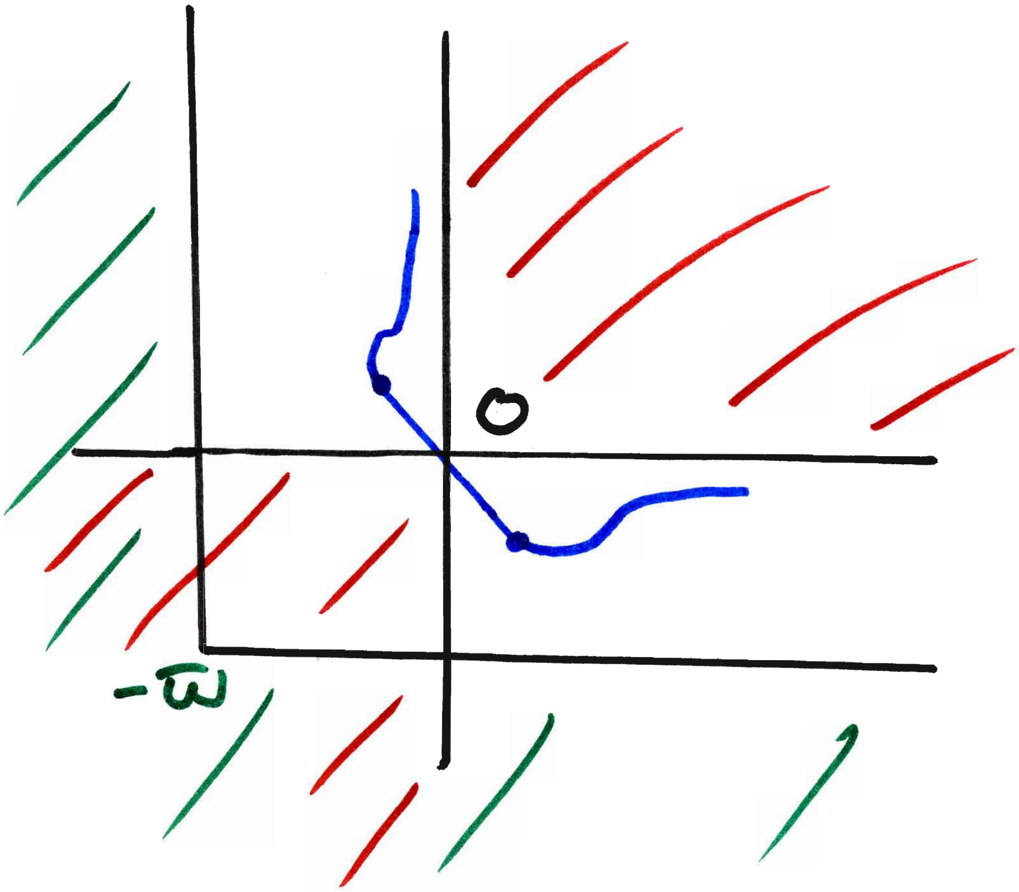
2. *for each individual* i*, the consumption set is* $\mathbf{R}_+^L$*, i.e. each good is perfectly divisible, and each agent is capable of surviving on zero consumption;*

3. *acyclicity: there is no collection* $x_1, x_2, \ldots, x_m$ *such that* $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m \succ_i x_1$*;*

Approximate Equilibrium with Nonconvexities



"offer curve"



Then there exists $p^* \gg 0$ and $x_i^* \in D_i(p^*)$ such that

$$\begin{aligned} &\frac{1}{I}\sum_{\ell=1}^{L} \max\left\{\left(\sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i\right)_\ell ,\ 0\right\} \\ &\leq\ 2\sqrt{\frac{L}{I}}\max\{\|\omega_i\|_1 : i = 1, \ldots, I\} \end{aligned} \tag{1}$$

where $\|x\|_1 = \sum_{\ell=1}^{L} |x_\ell|$.

Proof Outline:

– Let

$$\Delta' = \left\{p \in \mathbf{R}^L : \sqrt{\frac{L}{I}} \leq p_\ell \leq 1\ (\ell = 1, \ldots, L)\right\}$$

Why?

* Convenient to normalize prices by $\|p\|_\infty = 1$, would have to prove Kakutani's Theorem holds on that set, which is not convex. Δ' is convex and compact.
* In the proof of Debreu-Gale-Kuhn-Nikaido, we define the correspondence on Δ^0 and extend it to Δ. Get a fixed point $\hat{p}^* \in \Delta^0$, but we get no control over the price of the cheapest good. This gives us no control over the diameter of the budget set, which bounds the size of the nonconvexity in the demand set.

    * By trimming Δ' so all prices bounded below by $\sqrt{L/I}$, we give up the fact that the Kakutani fixed point is in the interior of Δ', but we get control over the diameter of the budget set.

- Consider the correspondence $z : \Delta' \to \mathbf{R}^L$ defined by

$$z(p) = \left(\sum_{i=1}^{I} \text{con } D_i(p) \right) - \bar{\omega}$$

- Acyclicity implies that $z(p) \neq \emptyset$.

- Look at the "Offer Curve"

$$\{x : \exists_{p \in \Delta'} \; x \in z(p)\}$$

  The picture is almost the same as the picture in the convex case.

- Choose a compact set $X \subset \mathbf{R}^L$ such that

$$p \in \Delta' \Rightarrow z(p) \subseteq X$$

  Define a correspondence $f : \Delta' \times X \to \Delta' \times X$ by

$$f(p, x) = \{(q, y) : y \in z(p), \; \forall_{q' \in \Delta'} \; q \cdot x \geq q' \cdot x\}$$

  By Kakutani's Theorem, there exists a fixed point $(p^*, \bar{x}^*)$

$$\begin{aligned} \bar{x}^* &= \sum_{i=1}^{I} \bar{x}_i^* \\ \bar{x}_i^* &\in \text{con } E_i(p^*) \ (i = 1, \ldots, I) \end{aligned} \tag{2}$$

$$\forall_{q \in \Delta'} \ q \cdot \bar{x}^* \leq p \cdot \bar{x}^* = 0 \tag{3}$$

– In the proof of Debreu-Gale-Kuhn-Nikaido, we showed that

$$\forall_{q \in \Delta} \ q \cdot \bar{x}^* \leq 0 \Rightarrow \bar{x}^* \leq 0$$

Use a similar argument and Equation (3) to put an upper bound on the positive components of $\bar{x}^*$.

– From Equation (2) and the Shapley-Folkman Theorem, we can assume that

$$\begin{aligned} \bar{x}_{i_\ell}^* &\in \text{con } E_{i_\ell}(p^*) \ (\ell = 1, \ldots, L) \\ \bar{x}_i^* &\in E_i(p^*) \text{ for } i \notin \{i_1, \ldots, i_L\} \end{aligned}$$

– Choose arbitrarily

$$x_{i_1}^* \in E_{i_1}(p^*), \ldots, x_{i_L}^* \in E_{i_L}(p^*)$$

and let

$$x_i^* = \bar{x}_i^* \text{ for } i \notin \{i_1, \ldots, i_L\}$$

so

$$x_i^* \in E_i(p^*) \text{ for all } i$$

–

$$\sum_{i=1}^{I} x_i^* = \bar{x}^* + \underbrace{\sum_{\ell=1}^{L} \left(x_{i_\ell}^* - \bar{x}_{i_\ell}^* \right)}_{\text{Error Term}}$$

– The diameters of the budget sets are bounded above by the endowments and the lower bound on prices in Δ', which bounds the Error Term.

Indivisibilities:

Theorem 5 applies verbatim to the case of indivisibilites, except that one must substitute Q_i for D_i. With indivisibilities, Q_i has closed graph but D_i generally does not.