

Lecture 13

Justifying (or Undermining) the Price-Taking Assumption

- Many formulations: Core, Ostroy’s No Surplus Condition, Bargaining Set, Shapley-Shubik Market Games (noncooperative), other noncooperative games
- Core is the most commonly used. The *core* is the set of all allocations such that no coalition (set of agents) can *improve on* or *block* the allocation (make all of its members better off) by seceding from the economy and only trading among its members.
- Core is institution-free; no mention of prices.
- “Core convergence” means roughly that

For economies with a large number of agents, core allocations are “approximately Walrasian.”

- “Approximately Walrasian” means different things in different contexts, depending on what we are willing to assume.
- *Three motivations for the study of the core:*

- *Walrasian allocations lie in the core*: Important strengthening of First Welfare Theorem, under same minimal assumptions as First Welfare Theorem.
 - * (Positive): Strong stability property of Walrasian equilibrium: no group of individuals would choose to upset the equilibrium by recontracting among themselves.
 - * (Normative): If distribution of initial endowments is equitable, no group is treated unfairly at a core allocation. Since Walrasian allocations lie in the core, this is a Group Fairness Property of Walrasian Equilibrium.

- *Core Convergence strengthens Second Welfare Theorem*
 - * Second Welfare Theorem says every Pareto Optimum is a Walrasian Equilibria with Transfers.
 - * Core convergence asserts that core allocations of large economies are *nearly* Walrasian *without* transfers.
 - * One version states that core allocations can be realized as *exact* Walrasian equilibrium with *small* income transfers.

- * *Strong “unbiasedness” property of Walrasian equilibrium*
 - Restricting to Walrasian outcomes does not narrow possible outcomes beyond narrowing occurring in the core.
 - (Normative) No hidden implications for welfare of different groups beyond equity issues in the initial endowment distribution.
 - (Normative) Assuming distribution of endowments is equitable, any allocation that is far from Walrasian will not be in the core, and hence will treat some group unfairly.
- *Core Convergence justifies Price-Taking, Core Nonconvergence suggests Price-Taking is Implausible:*
 - * The definition of Walrasian equilibrium contains (hidden in plain sight) assumption that economic agents act as price-takers.
 - * In real markets, we see prices used to equate supply and demand, but this does not guarantee Walrasian outcome.
 - * Agents possessing market power may choose to supply quantities different from the competitive supply for the prevailing price, thereby altering that price and leading to a non-Pareto Optimal outcome.

- * If outcome is not Walrasian, Welfare Theorems, Existence, Determinacy would have limited implications for real economies.
- * (Positive) Core convergence and nonconvergence allows us to identify situations in which price-taking is more or less reasonable.
- * Edgeworth defined core.
- * Edgeworth criticized Walras, thought the core, not the set of Walrasian equilibria, was best positive description of outcomes from market mechanism.
 - In particular, the definition of the core does not impose the assumption of price-taking behavior made by Walras.
 - Furthermore, if any allocation not in the core arose, some group would find it in its interests to recontract. Edgeworth thus argues that the core is the significant positive equilibrium concept.
 - If core is correct positive concept, core convergence justifies price-taking. Core convergence says all trade takes place at almost a single price. Agent who tries to bargain cannot influence prices much, and cannot change outcome much (argument more compelling with stronger convergence notions).

- If core is correct positive concept, core nonconvergence undermines price-taking. Edgeworth himself argued that in real life, the presence of large firms leads to failure of price-taking.

- **Definition 1** In an exchange economy, a *coalition* is a set

$$S \subseteq \{1, \dots, I\}$$

A coalition S *blocks* or *improves on* an exact allocation x by x' if

$$\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$$

and

$$\forall_{i \in S} x'_i \succ_i x_i$$

The *core* is the set of all exact allocations which cannot be improved on by any nonempty coalition.

- Notice we follow MWG and require $x'_i \succ_i x_i$ for all $i \in S$; this is analogous to the definition of weakly Pareto Optimal. *Natural*: status quo should be focal, need strict improvement to join a coalition to upset the status quo.

- **Theorem 2** *In an exchange economy, every core allocation is weakly Pareto Optimal.*

Proof: If x is not weakly Pareto Optimal, then there exists x' ,

$$\sum_{i=1}^I x'_i = \bar{\omega}, \quad x'_i \succ_i x_i$$

Then $S = \{1, \dots, I\}$ improves on x by x' , so x is not in the core. ■

- **Theorem 3 (Strong First Welfare Theorem)** *In an exchange economy, every Walrasian Equilibrium lies in the core.*

Proof: Suppose (p^*, x^*) is a Walrasian Equilibrium. If x^* is not in the core, there exists $S \subseteq I$,

$S \neq \emptyset$ and $x'_i (i \in S)$ such that

$$\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i, \quad \forall i \in S \quad x'_i \succ_i x_i^*$$

Since $x_i^* \in D_i(p^*)$,

$$p^* \cdot x'_i > p^* \cdot \omega_i$$

so

$$\begin{aligned} p^* \cdot \sum_{i \in S} x'_i &= \sum_{i \in S} p^* \cdot x'_i \\ &> \sum_{i \in S} p^* \cdot \omega_i \\ &= p^* \cdot \sum_{i \in S} \omega_i \end{aligned}$$

but

$$\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$$

contradiction. Therefore, x^* is in the core. ■

Theorem 4 *Suppose we are given an exchange economy with L commodities, I agents, and preferences \succ_1, \dots, \succ_I satisfying weak monotonicity (if $x \gg y$, then $x \succ_i y$) and the following free disposal condition:*

$$x \gg y, y \succ_i z \Rightarrow x \succ_i z.$$

If x is in the core, then there exists $p \in \Delta$ such that

$$\frac{1}{I} \sum_{i=1}^I |p \cdot (x_i - \omega_i)| \leq \frac{2L}{I} \max\{\|\omega_1\|_\infty, \dots, \|\omega_I\|_\infty\} \quad (1)$$

$$\frac{1}{I} \sum_{i=1}^I |\inf\{p \cdot (y - x_i) : y \succ_i x_i\}| \leq \frac{4L}{I} \max\{\|\omega_1\|_\infty, \dots, \|\omega_I\|_\infty\} \quad (2)$$

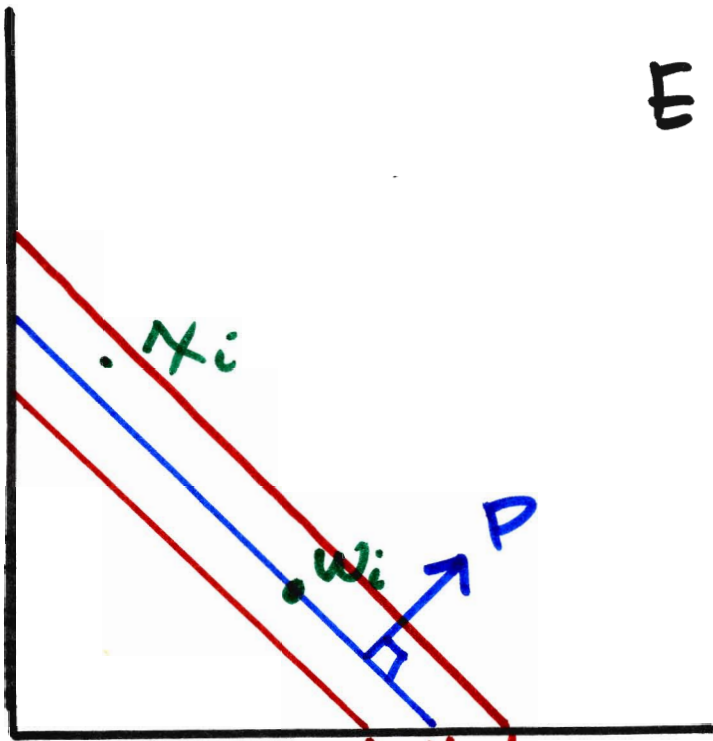
where $\|x\|_\infty = \max\{|x_1|, \dots, |x_L|\}$.

- Equation (1) says that trade occurs almost at the price p , and that each x_i is almost in the budget set.
- Equation (2) says that the price p almost supports \succ_i at x_i .
- If we knew the left sides of Equations (1) and (2) were *zero*, then

$$p \cdot (x_i - \omega_i) = 0 \Rightarrow x_i \in B_i(p)$$

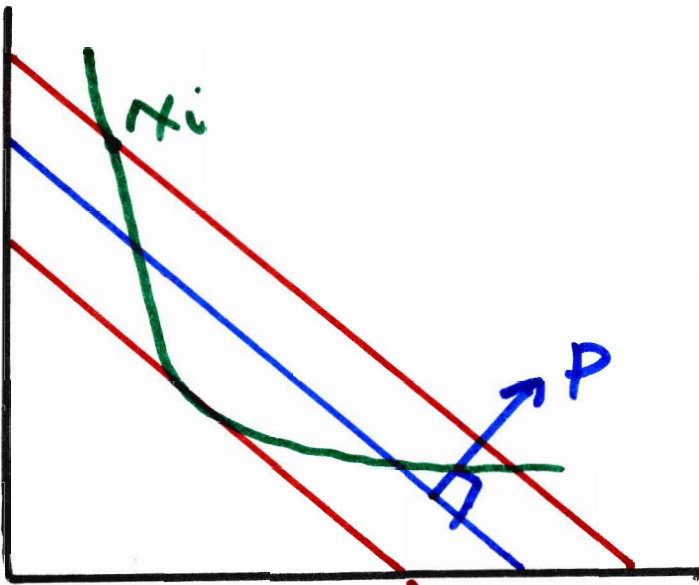
$$y \succ_i x_i \Rightarrow p \cdot y \geq p \cdot \omega_i$$

Equation (1)



$$\frac{2L}{I} \max \{ \|w_i\|_\infty : i=1, \dots, I \}$$

Equation (2)



$$\frac{4L}{I} \max \{ \|w_i\|_\infty : i=1, \dots, I \}$$

so x is a Walrasian quasiequilibrium! Thus, every core allocation satisfies a perturbation of the definition of Walrasian Equilibrium: agent i 's consumption need not lie in his/her budget set, but it can't be far outside; anything strictly preferred need not be outside the budget set, but it can't be far below the budget frontier.

Outline of Proof: Follow the proof of the Second Welfare Theorem.

- Suppose x is in the core. Define

$$\begin{aligned} B_i &= \{y - \omega_i : y \succ_i x_i\} \cup \{0\} \\ &= (\{y : y \succ_i x_i\} \cup \{\omega_i\}) - \omega_i \\ B &= \sum_{i=1}^I B_i \end{aligned}$$

The first term in the definition of B_i corresponds to members of a potential improving coalition; for accounting purposes, we assign members outside the coalition their endowments. Note that B_i is *not* convex, even if \succ_i is a convex preference.

- *Claim:* If x is in the core, then

$$B \cap \mathbf{R}_{--}^L = \emptyset$$

Suppose $z \in B \cap \mathbf{R}_{--}^L$. Then

$$\exists_{z_i \in B_i} z = \sum_{i=1}^I z_i$$

Let

$$S = \{i : z_i \neq 0\}$$

Since $z \ll 0$, $S \neq \emptyset$. For $i \in S$, let

$$x'_i = \omega_i + z_i - \frac{z}{|S|}$$

$$x'_i \gg \omega_i + z_i \succ_i x_i \text{ (definition of } B_i)$$

$$x'_i \succ_i x_i \text{ (free disposal)}$$

$$\begin{aligned} \sum_{i \in S} x'_i &= \sum_{i \in S} \omega_i + \sum_{i \in S} z_i - z \\ &= \sum_{i \in S} \omega_i + z - z \\ &= \sum_{i \in S} \omega_i \end{aligned}$$

so S can improve on x by x' , so x is not in the core.

• Let

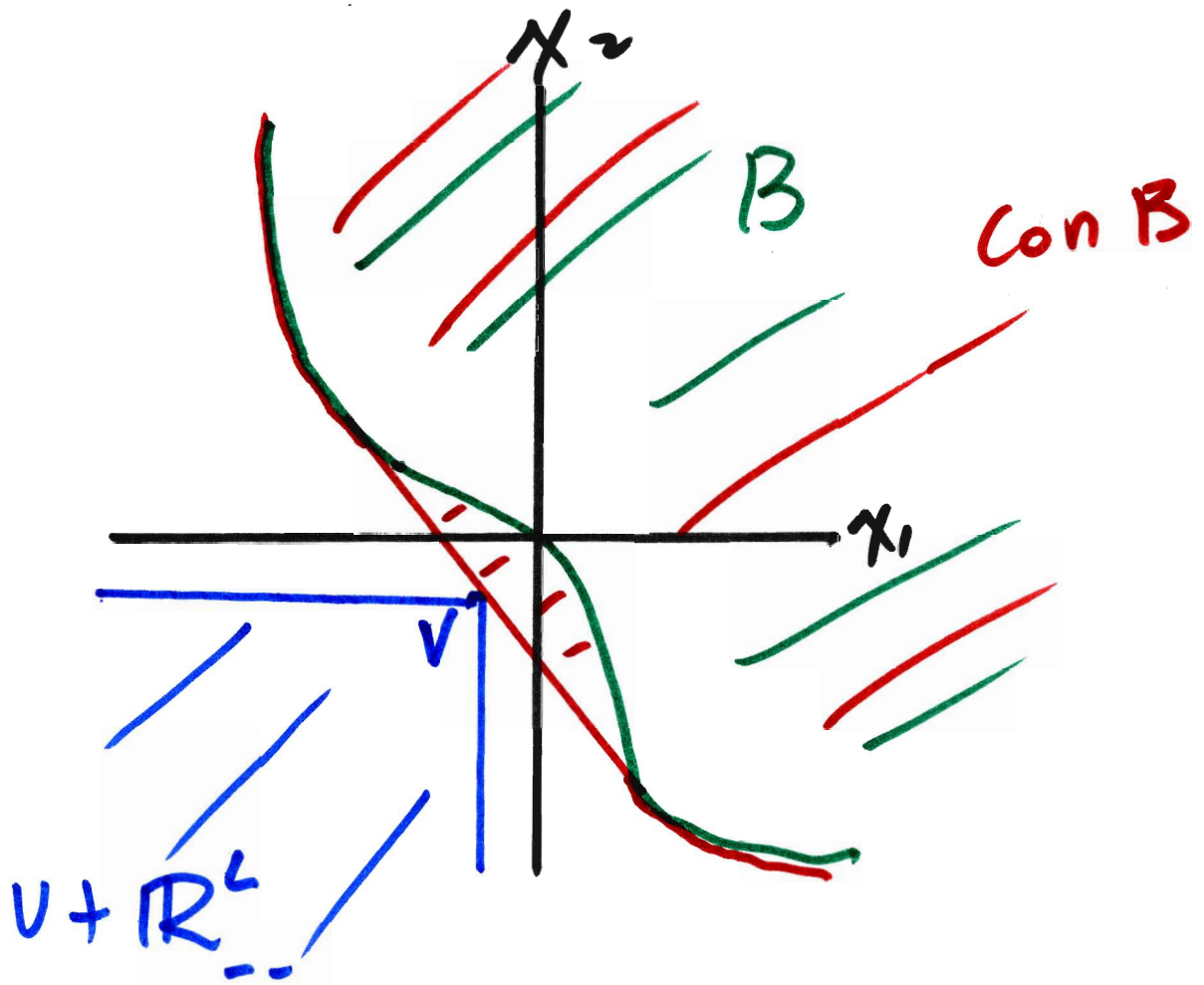
$$v = -L(\max_{i=1, \dots, I} \|\omega_i\|_\infty, \dots, \max_{i=1, \dots, I} \|\omega_i\|_\infty)$$

Claim:

$$(\text{con } B) \cap (v + \mathbf{R}_{--}^L) = \emptyset$$

If $z \in \text{con } B$, by the Shapley-Folkman Theorem, and relabelling the agents, we may write

$$z = \sum_{i=1}^I z_i, \quad z_i \in \text{con } B_i \text{ (} i = 1, \dots, I), \quad z_i \in B_i \text{ (} i \neq \{1, \dots, L\})$$



Choose

$$\hat{z}_i = \begin{cases} 0 & \text{if } i = 1, \dots, L \\ z_i & \text{if } i = L + 1, \dots, I \end{cases}$$

Then $\sum_{i=1}^I \hat{z}_i \in B$ so

$$\sum_{i=1}^I \hat{z}_i \not\ll 0$$

If $z \ll v$, then

$$\begin{aligned} \sum_{i=1}^I \hat{z}_i &= \sum_{i=1}^L 0 + \sum_{i=L+1}^I z_i \\ &\leq \sum_{i=1}^L (\omega_i + z_i) + \sum_{i=L+1}^I z_i \\ &= \sum_{i=1}^L \omega_i + \sum_{i=1}^I z_i \\ &= \sum_{i=1}^L \omega_i + z \\ &\ll \sum_{i=1}^L \omega_i + v \\ &\leq 0 \end{aligned}$$

so

$$B \cap \mathbf{R}_{--}^L \neq \emptyset$$

a contradiction which proves the claim.

- By Minkowski's Theorem, there exists $p \neq 0$ such that

$$\sup p \cdot (v + \mathbf{R}_{--}^L) \leq \inf p \cdot (\text{con } B)$$

If $p_\ell < 0$ for some ℓ , then

$$\sup p \cdot (z + \mathbf{R}_{--}^L) = +\infty$$

$$\inf p \cdot (\text{con } B) \leq 0$$

contradiction, so $p > 0$ and we can normalize $p \in \Delta$.

$$\inf p \cdot B \geq \inf p \cdot (\text{con } B)$$

$$\geq p \cdot v$$

$$= -L \max \{ \|\omega_1\|_\infty, \dots, \|\omega_I\|_\infty \}$$

- Adapt the remainder of the proof of the Second Welfare Theorem (requires a few tricks).