Economics 201B–Second Half
Lecture 2
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Two Graphical "Proofs" of the Existence of Walrasian Equilibrium in the Edgeworth Box

Demand: $D_i(p) = \{x \in B_i(p) : \forall_{y \in B_i(p)} x \succeq_i y\}$

Walrasian Equilibrium (in the Edgeworth Box) is a pair (p, x) where

- x is an exact allocation
- $x_i \in D_i(p)$ $(i = 1, 2)$

In the following Edgeworth Box Diagram, we give a graphical representation of Walrasian Equilibrium. In fact, there are (at least) three Walrasian Equilibria in the drawing, and there is nothing apparently pathological in the preferences of the two agents. **Note that if the demands of the two agents at a single price p are represented by the same point in the Edgeworth Box, it indicates that the sum of the demands equals the total supply, so we have Walrasian Equilibrium; on the other hand, if the demands of the two agents at a price p are represented by different points in the Edgeworth Box, the sum of the demands does *not* equal the total supply; p is not an equilibrium price.

Why the quotes on “Proofs”? Why the Proofs inside the quotes?

- graphical arguments prone to introduction of tacit assumptions
- these arguments can be turned into proofs; our real proof later follows the first of the two “proofs”

Price Normalization: $p \in \Delta^0 = \{p \in \mathbf{R}^2_{++} : p_1 + p_2 = 1\}$; $\Delta = \{p \in \mathbf{R}^2_{+} : p_1 + p_2 = 1\}$
Notation:

- $D(p) = D_1(p) + D_2(p)$ Market Demand
- $E_i(p) = D_i(p) - \omega_i$ Excess Demand of i
- $E(p) = E_1(p) + E_2(p) = D(p) - \bar{\omega}$ Market Excess Demand
- *Offer Curve:*
  - $OC_i = \{x : \exists_{p \in \Delta^0}\, x \in D_i(p)\}$ This is a curve in the Edgeworth Box Diagram; OC_1 measured from O_1, OC_2 from O_2.
  - $OC = \{x : \exists_{p \in \Delta^0}\, x \in E(p)\}$ This is a curve in $\mathbf{R}^2$.
  - $0 \in OC \Leftrightarrow$ there is a Walrasian Equilibrium: straightforward.

  - $(OC_1 \cap OC_2) \setminus \{\omega\} \neq \emptyset \Rightarrow$ there is a Walrasian Equilibrium; we'll see why.

Items Common to the two "Proofs:"

- **Lemma 1** *If* $p_n \in \Delta^0$ *and* $p_{n\ell} \to 0$ *as* $n \to \infty$, *then* $|D_i(p_n)| \to \infty$.

  This follows from strong monotonicity, and was likely proved in 201A. We'll prove later in a more general case.

- *Walras' Law:*
  - $p \cdot D_i(p) \leq p \cdot \omega_i$. Comes from definition, with no assumptions on preferences.
  - By strong monotonicity, can't have $p \cdot D_i(p) < p \cdot \omega_i$, so $p \cdot D_i(p) = p \cdot \omega_i$, so $p \cdot E_i(p) = 0$, so $p \cdot E(p) = 0$. In particular,

$$\begin{array}{c} \not\exists_{p \in \Delta^0} \ (D_i(p) < \omega_i \vee D_i(p) > \omega_i) \\ \not\exists_{p \in \Delta^0} \ (E(p) < 0 \vee E(p) > 0) \end{array} \tag{1}$$

- Observe that **this is where we use the fact that $D_i(p) \geq 0$, equivalently $E_i(p) \geq -\omega_i$:

$$\begin{aligned} (p_n)_2 D(p_n)_2 &\leq p_n \cdot D(p_n) \\ &= p_n \cdot \bar{\omega} \\ &\leq \max\{\bar{\omega}_1, \bar{\omega}_2\} \end{aligned}$$

If $p_{n1} \to 0$, $p_{n2} \to 1$, so for n sufficienly large, $D(p_n)_2 \leq 2\max\{\bar{\omega}_1, \bar{\omega}_2\}$, so $D(p_n)_2 \not\to \infty$. Therefore,

$$\begin{array}{l} p_{n1} \to 0 \Rightarrow D(p_n)_1 \to \infty \Rightarrow E(p_n)_1 \to \infty \\ p_{n2} \to 0 \Rightarrow D(p_n)_2 \to \infty \Rightarrow E(p_n)_2 \to \infty \end{array} \qquad (2)$$

- Given $p \in \Delta^0$, $D_1(p), D_2(p)$ and $E(p)$ each consist of a single element. In other words, every ray through the origin with negative slope intersects OC in exactly one point other than zero. In the Edgeworth Box diagram, each ray through ω with negative slope intersects OC_1 and OC_2 in exactly one point, other than ω, each. Given a point $x \in OC, x \neq 0$, there is a unique $p \in \Delta^0$ such that $x \in E(p)$; p is the perpendicular to the ray through 0 and x. Given a point $x \in OC_i$, $x \neq \omega$, there is a unique $p \in \Delta^0$ such that $x \in D_i(p)$; p is the perpendicular to the ray through ω and x.

"Proof 1:" (In Consumption Space, using OC)

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$$\begin{aligned} 0 \in OC &\Leftrightarrow \exists_{p \in \Delta^0} E(p) = 0 \\ &\Leftrightarrow \text{Walrasian Equilibrium exists} \end{aligned}$$

Hence, it suffices to show that $0 \in OC$

- $$E(p) = D(p) - \bar{\omega} \geq -\bar{\omega} \tag{3}$$

- In the following diagram, Equations (2) and (3) tell us that OC goes from the region (A) (when p_1 is small) to the region (B) (when p_2 is small).

- Equation (1) tells us that OC avoids the first (northeast) and third (southwest) quadrants, *so OC must pass through zero, so Walrasian Equilibrium exists!*

- However, it appears that OC may go through the origin more than once, reinforcing the earlier conclusion that Walrasian Equilibrium need not be unique.

"Proof 2" (uses OC_1 and OC_2 as in diagrams in MWG, assumes preferences are smooth)

- Suppose $x \in OC_1 \cap OC_2$, $x \neq \omega$. Then $x_i = D_i(p_i)$ for some $p_i \in \Delta^0$ $(i = 1, 2)$, so $x_i \geq 0$, and hence x lies in the Edgeworth Box; although each offer curve can go outside the Edgeworth Box, any intersection of the offer curves must lie in the Edgeworth Box.** There is a unique ray going through x and ω, and p_1 and p_2 are both perpendicular to it, so $p_1 = p_2$. Since x is a point in the Edgeworth Box, $x_1 + x_2 = \bar{\omega}$, so p_1 is a Walrasian Equilibrium Price. In other words, it suffices to show that $OC_1 \cap OC_2$ contains at least one $x \neq \omega$.

- $$\omega \in OC_1 \cap OC_2 \tag{4}$$
  To see this, let p_i be the "support price" to $\succeq_i$ at ω_i. In other words,
  $$y \succeq_i \omega_i \Rightarrow p_i \cdot y \geq p_i \cdot \omega_i$$
  We'll explain more carefully later why the support price exists. Then $\omega_i = D_i(p_i)$ so $\omega_i \in OC_i$, so $\omega \in OC_1 \cap OC_2$.
- **If preferences are smooth, then
  $$\begin{aligned} & p_i \cdot (D_i(p) - \omega_i) \\ & \quad = p \cdot (D_i(p) - \omega_i) + (p_i - p) \cdot (D_i(p) - D_i(p_i)) \\ & \quad = 0 \text{ (by Walras' Law) } + O(|p_i - p|^2) \end{aligned}$$
  which shows that p_i is tangent to OC_i at ω_i.
- If it turns out that $p_1 = p_2$, then this common price is a Walrasian Equilibrium Price and ω is a Walrasian Equilibrium allocation. If $p_1 \neq p_2$, then
  - OC_1 and OC_2 cross at ω.
  - By Equation (1), $OC_1 \cup OC_2$ cannot enter the quadrant northeast of ω or the quadrant southwest of ω.

- By Equation (2), as the price of the first good moves from 0 to 1, OC_1 and OC_2 travel from (A) to (B). Notice that OC_1 at (A) lies northeast of OC_2 at (B), and OC_1 at (B) lies northeast of OC_2 at (A). Thus, OC_1 and OC_2 "must" cross an even number of times, hence they cross at some $x \neq \omega$, so Walrasian Equilibrium exists.