Economics 201B–Second Half

Lecture 8

Existence of Walrasian Equilibrium (Continued)

Proposition 1 (17.C.1) Debreu-Gale-Kuhn-Nikaido Lemma *Suppose* $z : \Delta^0 \to \mathbf{R}^L$ *is a function satisfying*

1. *continuity*
2. *Walras' Law*
$$\forall_{p \in \Delta^0}\ p \cdot z(p) = 0$$
3. *bounded below:*
$$\exists_{x \in \mathbf{R}^L} \forall_{p \in \Delta^0}\ z(p) \geq x$$
4. *Boundary Condition: If* $p_n \to p$ *where* $p \in \Delta \setminus \Delta^0$*, then*
$$|z(p_n)| \to \infty$$

Then there exists $p^* \in \Delta^0$ *such that*

$$z(p^*) = 0$$

Outline of proof:

- Define a correspondence $f : \Delta^0 \to \Delta$ (so $f(p) \in 2^\Delta$) by

$$f(p) = \{q \in \Delta : q \cdot z(p) \geq q' \cdot z(p) \text{ for all } q' \in \Delta\}$$

  f identifies the goods in highest excess demand.

- Extend the domain of f to Δ to make it have closed graph.

- Verify that if $p^* \in f(p^*)$, then $p^* \in \Delta^0$ and $z(p^*) = 0$.

- Check that f satisfies the hypotheses of Kakutani's Theorem.

- By Kakutani's Theorem, there exists $p^* \in \Delta$ such that $p^* \in f(p^*)$, so $p^* \in \Delta^0$ and $z(p^*) = 0$.

Details of proof:

- Define a correspondence $f : \Delta^0 \to \Delta$ (so $f(p) \in 2^\Delta$) by

$$f(p) = \{q \in \Delta : q \cdot z(p) \geq q' \cdot z(p) \text{ for all } q' \in \Delta\}$$

f identifies the goods in highest excess demand.

$$\forall_{\ell \neq \ell_0}\ z(p)_{\ell_0} > z(p)_\ell \Rightarrow$$

$$f(p) = \{(0, \ldots, 0, \underset{\substack{\uparrow \\ \ell_0}}{1}, 0, \ldots, 0)\}$$

$$z(p)_{\ell_0} = z(p)_{\ell_1} > z(p)_\ell \text{ for all } \ell \notin \{\ell_0, \ell_1\} \Rightarrow$$

$$f(p) = \{(0, \ldots, 0, \underset{\substack{\uparrow \\ \ell_0}}{\alpha}, 0, \ldots, 0, \underset{\substack{\uparrow \\ \ell_1}}{\mathbf{1} - \alpha}, 0, \ldots, 0) : \alpha \in [0,1]\}$$

- Extend the domain of f to Δ to make it have closed graph. For $p \in \Delta \setminus \Delta^0$, let

  $$\begin{aligned} f(p) &= \{q \in \Delta : p \cdot q = 0\} \\ &= \{q \in \Delta : p_\ell > 0 \Rightarrow q_\ell = 0\} \end{aligned}$$

  We will verify f has closed graph on Δ in the fourth step.

- Verify that if $p^* \in f(p^*)$, then $p^* \in \Delta^0$ and $z(p^*) = 0$.
  - We claim that

    $$p^* \in \Delta^0$$

If $p^* \in \Delta \setminus \Delta^0$, then

$$\begin{aligned} & \forall_{q \in f(p^*)}\ p^* \cdot q = 0 \text{ (definition of } f) \\ \Rightarrow\ & p^* \cdot p^* = 0 \text{ (since } p^* \in f(p^*)) \\ \Rightarrow\ & p^* = 0 \\ \Rightarrow\ & p^* \notin \Delta \end{aligned}$$

contradiction. Therefore,

$$p^* \in \Delta^0$$

- We claim that

  $$p^* \in f(p^*),\ p^* \in \Delta^0 \Rightarrow z(p^*) = 0$$

  We can't have $z(p^*) < 0$, for then $p^* \cdot z(p^*) < 0$, contradicting Walras' Law. Fix $\ell \in \{1, \ldots, L\}$

  Let

  $$\begin{aligned} q &= (0, \ldots, 0, \underset{\substack{\uparrow \\ \ell}}{1}, 0, \ldots, 0)\} \\ z(p^*)_\ell &= q \cdot z(p^*) \\ &\leq p^* \cdot z(p^*) \ (p^* \in f(p^*),\ \text{definition of } f) \\ &= 0 \text{ (Walras' Law)} \end{aligned}$$

  Therefore, $z(p^*) \leq 0$ but $z(p^*) \not< 0$, so

  $$z(p^*) = 0$$

- Check that f satisfies the hypotheses of Kakutani's Theorem.

  - Δ is a compact convex nonempty subset of $\mathbf{R}^L$.

- $f : \Delta \to \Delta$ is
    * *nonempty-valued*: If $p \in \Delta^0$,

      $$f(p) = \{q \in \Delta : \forall_{q' \in \Delta}\ q \cdot z(p) \geq q' \cdot z(p)\}$$

      $q \cdot z(p)$ is a continuous function of $q \in \Delta$, which is compact, so the function achieves its maximum, so $f(p) \neq \emptyset$.

      If $p \in \Delta \setminus \Delta^0$,

      $$f(p) = \{q \in \Delta : q \cdot p = 0\}$$

      Since $p \in \Delta \setminus \Delta^0$, $p_\ell = 0$ for some ℓ, so if we let

      $$q = (0, \ldots, 0, \underset{\substack{\uparrow \\ \ell}}{1}, 0, \ldots 0)$$

      then $q \in \Delta$ and $q \cdot p = 0$, so $f(p) \neq \emptyset$.

- *convex-valued*: Suppose $q, \hat{q} \in f(p)$, $\alpha \in (0,1)$. Since Δ is convex,

  $$\alpha q + (1-\alpha)\hat{q} \in \Delta$$

  If $p \in \Delta^0$, and $q' \in \Delta$,

  $$\begin{aligned}
  (\alpha q + (1-\alpha)\hat{q}) \cdot z(p) &= \alpha q \cdot z(p) + (1-\alpha)\hat{q} \cdot z(p) \\
  &\geq \alpha q' \cdot z(p) + (1-\alpha) q' \cdot z(p) \\
  &\quad \text{(definition of } f;\ q, \hat{q} \in f(p)\text{)} \\
  &= q' \cdot z(p)
  \end{aligned}$$

so

$$\alpha q + (1-\alpha)\hat{q} \in f(p)$$

If $p \in \Delta \setminus \Delta^0$,

$$\begin{aligned}
(\alpha q + (1-\alpha)\hat{q}) \cdot p &= \alpha q \cdot p + (1-\alpha)\hat{q} \cdot p \\
&= \alpha 0 + (1-\alpha) 0 \\
&\qquad \text{(definition of } f;\ q, \hat{q} \in f(p)) \\
&= 0
\end{aligned}$$

so

$$\alpha q + (1-\alpha)\hat{q} \in f(p)$$

– *upper hemicontinuous*: By Theorem 3 in Lecture 7, since Δ is compact, it is enough to show that f has closed graph. Suppose $p_n \to p$, $q_n \in f(p_n)$, and $q_n \to q$. We need to show that

$$q \in f(p)$$

If $p \in \Delta^0$, then $p_n \in \Delta^0$ for n sufficiently large, so

$$f(p_n) = \{q \in \Delta : \forall_{q' \in \Delta}\ q \cdot z(p_n) \geq q' \cdot z(p_n)\}$$

z is continuous on Δ^0, so

$$z(p_n) \to z(p)$$

Suppose $q' \in \Delta$.

$$\begin{aligned} q' \cdot z(p) &= q' \cdot \lim_{n\to\infty} z(p_n) \\ &= \lim_{n\to\infty} q' \cdot z(p_n) \\ &\leq \lim_{n\to\infty} q_n \cdot z(p_n) \\ &\leq \lim_{n\to\infty} q_n \cdot \lim_{n\to\infty} z(p_n) \\ &= q \cdot z(p) \end{aligned}$$

so

$$q \in f(p)$$

If $p \in \Delta \setminus \Delta^0$, may have $p_n \in \Delta^0$ for some n and $p_n \in \Delta \setminus \Delta^0$ for other n. We are in one or both of the following cases:

∗ *Case I*: There is a subsequence p_{n_k} such that $p_{n_k} \in \Delta^0$ for all k. In this case, we may assume WLOG that $p_n \in \Delta^0$ for all n. We need to show that $p \cdot q = 0$. Suppose $p_{\ell_0} > 0$; let $\alpha = \frac{p_{\ell_0}}{2}$. For n sufficiently large,

$$(p_n)_{\ell_0} \geq \alpha$$

$|z(p_n)| \to \infty$, and $z(p_n)$ is bounded below, so

$$\exists_{\ell_n \in \{1,\ldots,L\}} \; z(p_n)_{\ell_n} \to \infty$$

But

$$\begin{aligned} (p_n)_{\ell_0} z(p_n)_{\ell_0} &= p_n \cdot z(p_n) - \sum_{\ell \neq \ell_0} (p_n)_\ell z(p_n)_\ell \\ &= -\sum_{\ell \neq \ell_0} (p_n)_\ell z(p_n)_\ell \\ &\leq \|x\|_\infty \\ z(p_n)_{\ell_0} &\leq \frac{\|x\|_\infty}{\alpha} \end{aligned}$$

so for n sufficiently large,

$$\begin{aligned} z(p_n)_{\ell_0} < z(p_n)_{\ell_n} &\Rightarrow (q_n)_{\ell_0} = 0 \\ &\Rightarrow q_{\ell_0} = 0 \end{aligned}$$

Therefore,

$$p_{\ell_0} > 0 \Rightarrow q_{\ell_0} = 0$$

so $q \cdot p = 0$ and $q \in f(p)$.

- $*$ *Case II*: There is a subsequence p_{n_k} such that $p_{n_k} \in \Delta \setminus \Delta^0$ for all k. In this case, we may assume WLOG that $p_n \in \Delta \setminus \Delta^0$ for all n. Then $q_n \cdot p_n = 0$ for all n, so

$$\begin{aligned} q \cdot p &= \left(\lim_{n\to\infty} q_n\right) \cdot \left(\lim_{n\to\infty} p_n\right) \\ &= \lim_{n\to\infty} q_n \cdot p_n \\ &= \lim_{n\to\infty} 0 \\ &= 0 \end{aligned}$$

so

$$q \in f(p)$$

- By Kakutani's Theorem, there exists $p^* \in \Delta$ such that $p^* \in f(p^*)$, so $p^* \in \Delta^0$ and $z(p^*) = 0$.■