Economics 201B–Second Half
Lecture 9
Revised 4/23/09, Revisions Indicated by * and Sticky Notes

Existence of Walrasian Equilibrium (Wrap-Up)

- What happens if we weaken the strong monotonicity assumption?
  - local nonsatiation implies Walras' Law holds with equality, but is not sufficient to give Walrasian Equilibrium with $\Sigma_{i=1}^{I} x_i^* \leq \bar{\omega}$.
    * In Edgeworth Box Economy, let

$$\begin{aligned} u_1(x,y) &= y + \sqrt{x} \text{ (strongly monotonic)} \\ \omega_1 &= (0,1) \\ u_2(x,y) &= \min\{x,y\} \text{ (weakly monotonic)} \\ \omega_2 &= (1,1) \end{aligned}$$

For any $p \gg 0$,

$$\begin{aligned} D_2(p) = (1,1) &= \omega_2 \\ D_1(p)_1 &> \omega_{11} \end{aligned}$$

For $p = (1,0)$ or $p = (0,1)$,

$$D_1(p) = \emptyset$$

But notice for $p = (1, 0)$

$$\omega_1 \in Q_1(p)$$
$$\omega_2 \in Q_2(p)$$

so $(1, 0)$ is a Walrasian Quasi-Equilibrium Price.

- Even without local nonsatiation,

$$\exists_{p^* \in \Delta,\, x_i^* \in Q_i(p^*)} \sum_{i=1}^{I} x_i^* \leq \bar{\omega}$$

Walrasian Quasi-Equilibrium exists, some goods may be left over; local nonsatiation does not imply allocation is exact, since some prices may be zero.

- If *one* agent (WLOG agent 1) is strongly monotonic and $\omega_1 \gg 0$, then $p^* \gg 0$, so

$$x_i^* \in D_i(p^*) \ (i = 1, \ldots, I)$$
$$\sum_{i=1}^{I} x_i^* \leq \bar{\omega}$$

If, in addition, all agents exhibit local nonsatiation,

$$\sum_{i=1}^{I} x_i^* = \bar{\omega}$$

- If $\omega_i \gg 0$ for all i,

$$p^* \cdot \omega_i > 0$$
$$x_i^* \in D_i(p^*)$$
$$\sum_{i=1}^{I} x_i^* \leq \bar{\omega}$$

Local nonsatiation need not imply allocation exact, since some prices may be zero.

- With nonconvex preferences or indivisibilities, see Lecture 12.

Generic Local Uniqueness of Equilibrium

Comparative Statics: In what direction does the equilibrium move if the underlying parameters of the economy change?

*A *Foundation for Comparative Statics*:

1. *Local Uniqueness*: For every equilibrium price $p^* \in \Delta$, there exists $\delta > 0$ such that there is no equilibrium price $q^* \in \Delta$ such that

$$q^* \neq p^*, \; |q^* - p^*| < \delta$$

2. For a sufficiently small change in the parameters of the economy, the number of equilibria is unchanged and each equilibrium moves

$$\begin{pmatrix} \text{continuously} \\ \text{differentiably} \end{pmatrix}$$

as the parameter changes.

Remark 1 **If local uniqueness fails, lattice-theoretic methods may still allow us to establish comparative statics results. We can look at an equilibrium correspondence and we may be able to say that the *set* of equilibria moves in a particular direction in response to a change in the underlying parameters (Milgrom, Shannon, others).

Two-Good Economy: Consider a 2-good economy, normalized prices $p \in \Delta^0$,

$$z(p) = \sum_{i=1}^{I} D_i(p) - \bar{\omega}$$

- Walras' Law with Equality implies that

  $$z(p)_1 = 0 \Rightarrow z(p)_2 = 0$$

  so we can capture the situation in a diagram in $\mathbf{R}^2$; let

  $$\hat{z}(p_1) = z(p_1, 1 - p_1)_1$$

  and plot $\hat{z}$ as a function of p_1.

- In Diagram I, a small shift in $\hat{z}$ results in a small shift of the equilibrium price; comparative statics are locally meaningful. Notice that

  $$\hat{z}(p_1) = 0 \Rightarrow \hat{z}'(p_1) \neq 0$$

  $\hat{z}$ cuts cleanly through 0, so we expect an odd number of equilibria.

- In Diagram II, there are two equilibria p_L^* and p_R^*.

- A small shift in $\hat{z}$ results in a small shift in p_L^*
- A small upward shift in $\hat{z}$ causes p_R^* to split in two; one moves left, the other moves right, so no local comparative statics.
- A small downward shift in $\hat{z}$ cause p_R^* to disappear!
- Notice that

$$\hat{z}(p_R^*) = 0 \text{ but } \hat{z}'(p_R^*) = 0$$

- Diagram III shows we could even have a whole interval of equilibrium prices. A small change in $\hat{z}$ results in a *discontinuous* shift in equilibrium price.

Multi-Good Case:

- Normalize $p_L = 1$ rather than $p \in \Delta$. Price is represented by

$$\hat{p} = (p_1, \ldots, p_{L-1}) \in \mathbf{R}_{++}^{L-1}$$

- Let

$$\hat{z}(\hat{p}) = (z_1(\hat{p}, 1), \ldots, z_{L-1}(\hat{p}, 1))$$

- Walras' Law with Equality and $p \gg 0$ implies that

$$\hat{z}(\hat{p}) = 0 \Leftrightarrow z(\hat{p}, 1) = 0$$

- Observe that

$$\hat{z} : \mathbf{R}_{++}^{L-1} \rightarrow \mathbf{R}^{L-1}$$

so $D\hat{z}$, the Jacobian matrix of $\hat{z}$, is $(L-1)\times(L-1)$

- *Definition*: An equilibrium price p^* is *regular* if

$$\det D\hat{z}|_{\hat{p}^*} \neq 0$$

This is equivalent to

$$\text{rank } Dz|_{p^*} = L-1$$

A *regular economy* is an economy for which every equilibrium price is regular.

- *Maintained Hypotheses for Remainder of 17.D*:
  - z satisfies the hypotheses of the Debreu-Gale-Kuhn-Nikaido Lemma.
  - z is homogeneous of degree zero, i.e.

$$\forall_{p\in\mathbf{R}^L_{++},\,\lambda>0} z(\lambda p) = z(p)$$

    Caution: $\hat{z}$ is *not* homogeneous because it is a representation of a normalized price ($p_L = 1$).
  - z is C^1. This appears technical, but it's strong and has economic consequences because it rules out boundary consumptions: demand necessarily has a kink at the price where demand first hits boundary. This can be weakened to allow boundary consumptions, and the theorems more or less hold.

Proposition 2 (17.D.1) *In a regular economy, (normalized) Walrasian Equilibrium prices are locally unique, and there are only finitely many equilibria.*

Proof: Suppose $\hat{z}(\hat{p}^*) = 0$. Since the economy is regular, $D\hat{z}|_{\hat{p}^*}$ is nonsingular. By the Inverse Function Theorem, there is a neighborhood U of $\hat{p}^*$ and a neighborhood V of 0 and a C^1 function $h : V \to U$, h is one-to-one and onto such that

$$\forall_{v \in V}\ \hat{z}(h(v)) = v,\ \forall_{u \in U}\ h(\hat{z}(u)) = u$$

If $u \in U$ and $\hat{z}(u) = 0$,

$$\begin{aligned} u &= h(\hat{z}(u)) \\ &= h(0) \\ \hat{p}^* &= h(\hat{z}(\hat{p}^*)) \\ &= h(0) \end{aligned}$$

Since h is a function, $u = \hat{p}^*$, so Equilibrium is locally unique.

Now, we show that there are a finite number of equilibria.

Claim: There is a compact set $\hat{K} \subset \mathbf{R}^{L-1}_{++}$ such that if $\hat{z}(\hat{p}^*) = 0$, then $\hat{p}^* \in ** \hat{K}$.

Suppose the claim is not true. Define $\phi : \mathbf{R}^{L-1}_{++} \to \Delta^0$ by

$$\phi(\hat{p}) = \frac{(\hat{p}, 1)}{|\hat{p}_1 + \ldots + \hat{p}_{L-1} + 1|}$$

Observe that ϕ is one-to-one and onto, is continuous, and has continuous inverse. Let

$$K_n = \left\{ p \in \Delta^0 : p_\ell \geq \frac{1}{n} \, (1 \leq \ell \leq L) \right\}$$

Then $\phi^{-1}(K_n)$ is the continuous image of a compact set, hence a compact subset of $\mathbf{R}^{L-1}_{++}$, so the set of equilibrium prices cannnot be contained in $\phi^{-1}(K_n)$. Thus, we may find a sequence $\hat{p}^*_n$ of equilibrium prices such that $\phi(\hat{p}^*_n) \notin K_n$, hence

$$\phi(\hat{p}^*_n) \to p \in \Delta \setminus \Delta^0$$

By the Boundary Condition,

$$|z(\phi(\hat{p}^*_n))| \to \infty$$

but

$$\begin{aligned} z(\phi(\hat{p}^*_n)) &= z(\hat{p}^*_n, 1) \text{ (homogeneity of degree zero)} \\ &= 0 \text{ (since } \hat{z}(\hat{p}^*_n) = 0) \end{aligned}$$

a contradiction which proves the claim.

Since $\hat{z}$ is continuous,

$$E = \left\{ \hat{p} \in \mathbf{R}^{L-1}_{++} : \hat{z}(\hat{p}) = 0 \right\}$$

$$= \hat{z}^{-1}(\{0\})$$
$$= \mathbf{R}^{L-1}_{++} \setminus \hat{z}^{-1}\left(\mathbf{R}^{L-1} \setminus \{0\}\right)$$

is closed; since E is a closed subset of the compact set K, E is compact. For each $\hat{p} \in E$, we may find $\delta_{\hat{p}} > 0$ such that

$$E \cap B\left(\hat{p}, \delta_{\hat{p}}\right) = \{\hat{p}\}$$

The collection

$$\left\{B\left(\hat{p}, \delta_{\hat{p}}\right) : \hat{p} \in E\right\}$$

is an open cover of E, hence has a finite subcover. Since each element of this finite subcover contains exactly one element of E, E is finite. ∎