

Lecture 9

Existence of Walrasian Equilibrium (Wrap-Up)

- What happens if we weaken the strong monotonicity assumption?
  - local nonsatiation implies Walras' Law holds with equality, but is not sufficient to give Walrasian Equilibrium with  $\sum_{i=1}^I x_i^* \leq \bar{\omega}$ .

\* In Edgeworth Box Economy, let

$$u_1(x, y) = y + \sqrt{x} \text{ (strongly monotonic)}$$

$$\omega_1 = (0, 1)$$

$$u_2(x, y) = \min\{x, y\} \text{ (weakly monotonic)}$$

$$\omega_2 = (1, 1)$$

For any  $p \gg 0$ ,

$$D_2(p) = (1, 1) = \omega_2$$

$$D_1(p)_1 > \omega_{11}$$

For  $p = (1, 0)$  or  $p = (0, 1)$ ,

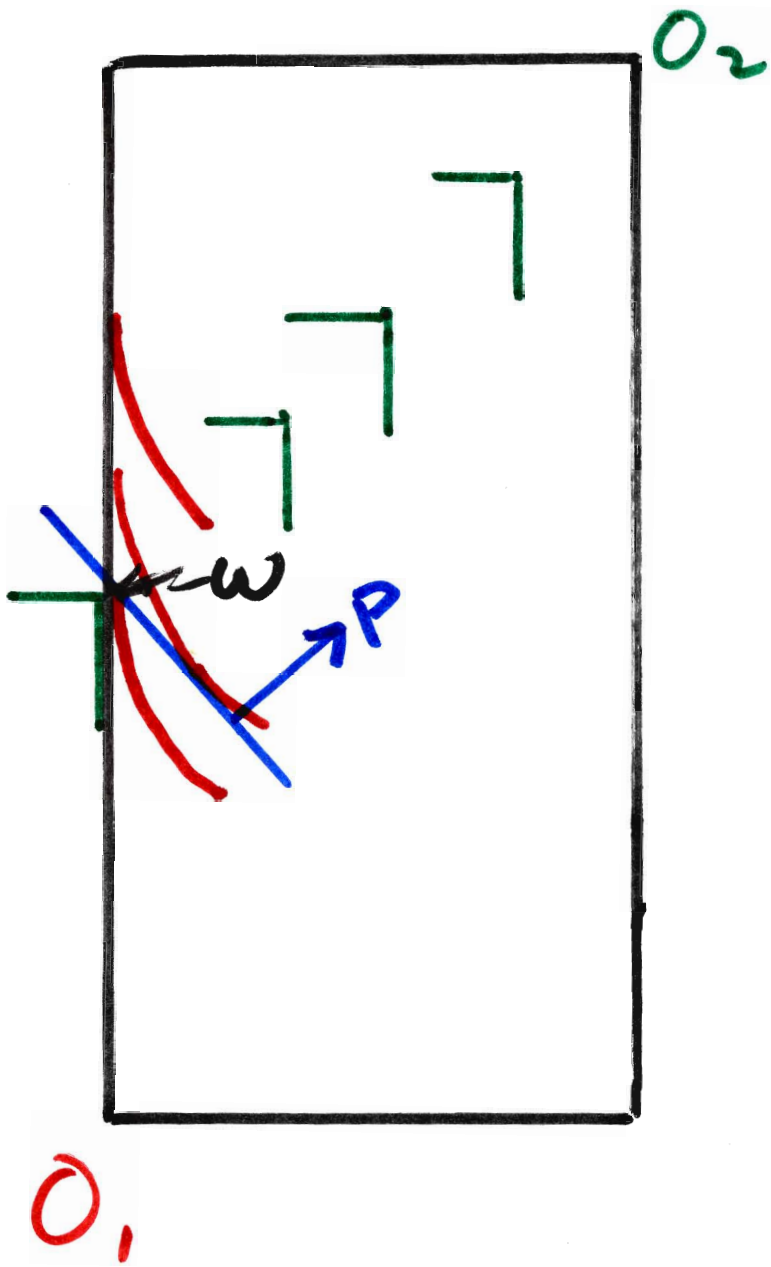
$$D_1(p) = \emptyset$$

But notice for  $p = (1, 0)$

$$\omega_1 \in Q_1(p)$$

$$\omega_2 \in Q_2(p)$$

so  $(1, 0)$  is a Walrasian Quasi-Equilibrium Price.



- Even without local nonsatiation,

$$\exists_{p^* \in \Delta, x_i^* \in Q_i(p^*)} \sum_{i=1}^I x_i^* \leq \bar{\omega}$$

Walrasian Quasi-Equilibrium exists, some goods may be left over; local nonsatiation does not imply allocation is exact, since some prices may be zero.

- If *one* agent (WLOG agent 1) is strongly monotonic and  $\omega_1 \gg 0$ , then  $p^* \gg 0$ , so

$$x_i^* \in D_i(p^*) \quad (i = 1, \dots, I)$$

$$\sum_{i=1}^I x_i^* \leq \bar{\omega}$$

If, in addition, all agents exhibit local nonsatiation,

$$\sum_{i=1}^I x_i^* = \bar{\omega}$$

- If  $\omega_i \gg 0$  for all  $i$ ,

$$p^* \cdot \omega_i > 0$$

$$x_i^* \in D_i(p^*)$$

$$\sum_{i=1}^I x_i^* \leq \bar{\omega}$$

Local nonsatiation need not imply allocation exact, since some prices may be zero.

- With nonconvex preferences or indivisibilities, see Lecture 12.

## Generic Local Uniqueness of Equilibrium

*Comparative Statics:* In what direction does the equilibrium move if the underlying parameters of the economy change?

*Minimal Foundation for Comparative Statics:*

1. *Local Uniqueness:* For every equilibrium price  $p^* \in \Delta$ , there exists  $\delta > 0$  such that there is no equilibrium price  $q^* \in \Delta$  such that

$$q^* \neq p^*, |q^* - p^*| < \delta$$

2. For a sufficiently small change in the parameters of the economy, the number of equilibria is unchanged and each equilibrium moves

$$\left( \begin{array}{c} \text{continuously} \\ \text{differentiably} \end{array} \right)$$

as the parameter changes.

*Two-Good Economy:* Consider a 2-good economy, normalized prices  $p \in \Delta^0$ ,

$$z(p) = \sum_{i=1}^I D_i(p) - \bar{\omega}$$

- Walras' Law with Equality implies that

$$z(p)_1 = 0 \Rightarrow z(p)_2 = 0$$

so we can capture the situation in a diagram in  $\mathbf{R}^2$ ; let

$$\hat{z}(p_1) = z(p_1, 1 - p_1)_1$$

and plot  $\hat{z}$  as a function of  $p_1$ .

- In Diagram I, a small shift in  $\hat{z}$  results in a small shift of the equilibrium price; comparative statics are locally meaningful. Notice that

$$\hat{z}(p_1) = 0 \Rightarrow \hat{z}'(p_1) \neq 0$$

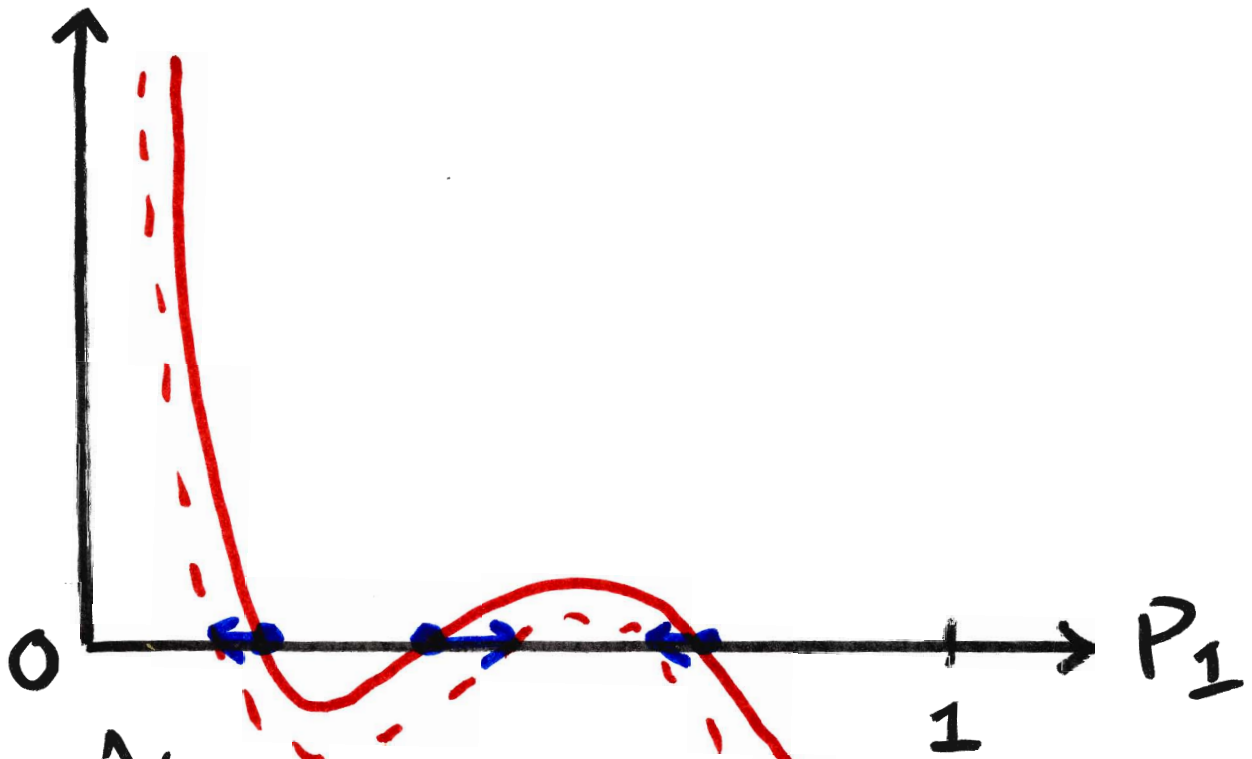
$\hat{z}$  cuts cleanly through 0, so we expect an odd number of equilibria.

- In Diagram II, there are two equilibria  $p_L^*$  and  $p_R^*$ .
  - A small shift in  $\hat{z}$  results in a small shift in  $p_L^*$
  - A small upward shift in  $\hat{z}$  causes  $p_R^*$  to split in two; one moves left, the other moves right, so no local comparative statics.
  - A small downward shift in  $\hat{z}$  cause  $p_R^*$  to disappear!
  - Notice that

$$\hat{z}(p_R^*) = 0 \text{ but } \hat{z}'(p_R^*) = 0$$

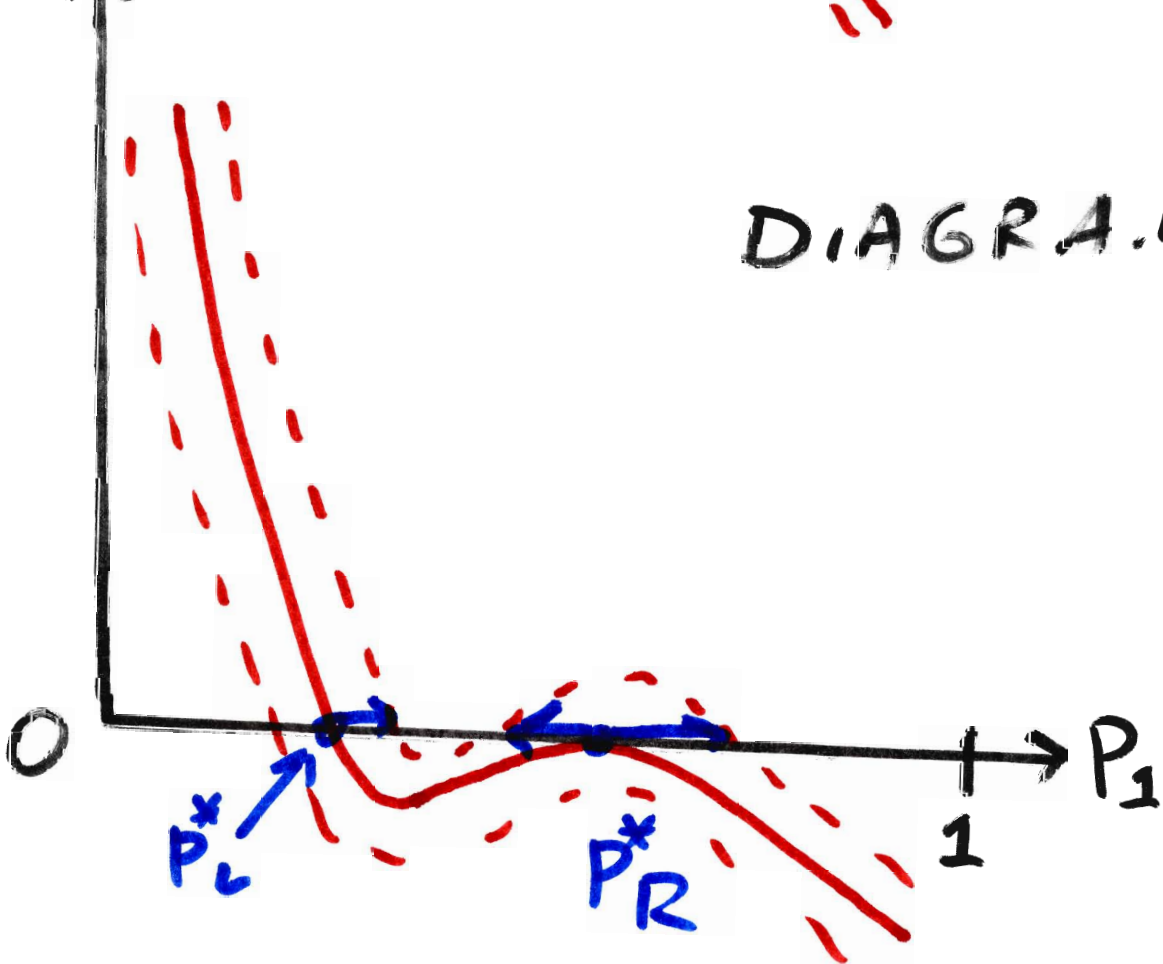
$\hat{z}(P_1)$

DIAGRAM I

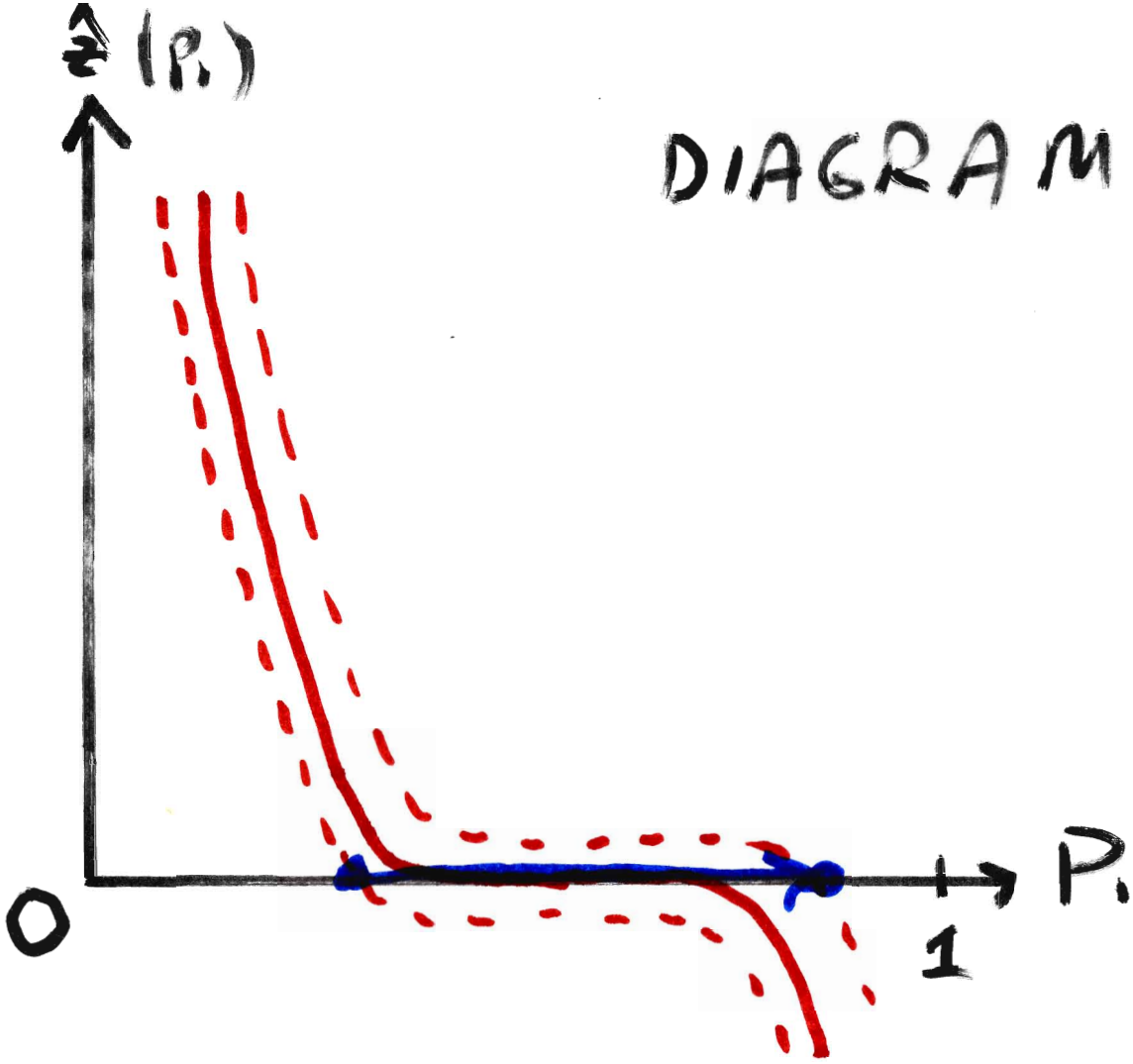


$\hat{z}(P_1)$

DIAGRAM II



# DIAGRAM III



- Diagram III shows we could even have a whole interval of equilibrium prices. A small change in  $\hat{z}$  results in a *discontinuous* shift in equilibrium price.

*Multi-Good Case:*

- Normalize  $p_L = 1$  rather than  $p \in \Delta$ . Price is represented by

$$\hat{p} = (p_1, \dots, p_{L-1}) \in \mathbf{R}_{++}^{L-1}$$

- Let

$$\hat{z}(\hat{p}) = (z_1(\hat{p}, 1), \dots, z_{L-1}(\hat{p}, 1))$$

- Walras' Law with Equality and  $p \gg 0$  implies that

$$\hat{z}(\hat{p}) = 0 \Leftrightarrow z(\hat{p}, 1) = 0$$

- Observe that

$$\hat{z} : \mathbf{R}_{++}^{L-1} \rightarrow \mathbf{R}^{L-1}$$

so  $D\hat{z}$ , the Jacobian matrix of  $\hat{z}$ , is  $(L-1) \times (L-1)$

- *Definition:* An equilibrium price  $p^*$  is *regular* if

$$\det D\hat{z}|_{\hat{p}^*} \neq 0$$

This is equivalent to

$$\text{rank } Dz|_{p^*} = L-1$$

A *regular economy* is an economy for which every equilibrium price is regular.

- *Maintained Hypotheses for Remainder of 17.D:*

- $z$  satisfies the hypotheses of the Debreu-Gale-Kuhn-Nikaido Lemma.
- $z$  is homogeneous of degree zero, i.e.

$$\forall_{p \in \mathbf{R}_{++}^L, \lambda > 0} z(\lambda p) = z(p)$$

Caution:  $\hat{z}$  is *not* homogeneous because it is a representation of a normalized price ( $p_L = 1$ ).

- $z$  is  $C^1$ . This appears technical, but it's strong and has economic consequences because it rules out boundary consumptions: demand necessarily has a kink at the price where demand first hits boundary. This can be weakened to allow boundary consumptions, and the theorems more or less hold.

**Proposition 1 (17.D.1)** *In a regular economy, (normalized) Walrasian Equilibrium prices are locally unique, and there are only finitely many equilibria.*

**Proof:** Suppose  $\hat{z}(\hat{p}^*) = 0$ . Since the economy is regular,  $D\hat{z}|_{\hat{p}^*}$  is nonsingular. By the Inverse Function Theorem, there is a neighborhood  $U$  of  $\hat{p}^*$  and a neighborhood  $V$  of 0 and a  $C^1$  function  $h : V \rightarrow U$ ,  $h$  is one-to-one and onto such that

$$\forall_{v \in V} \hat{z}(h(v)) = v, \forall_{u \in U} h(\hat{z}(u)) = u$$

If  $u \in U$  and  $\hat{z}(u) = 0$ ,

$$u = h(\hat{z}(u))$$

$$= h(0)$$

$$\hat{p}^* = h(\hat{z}(\hat{p}^*))$$

$$= h(0)$$

Since  $h$  is a function,  $u = \hat{p}^*$ , so Equilibrium is locally unique.

Now, we show that there are a finite number of equilibria.

*Claim:* There is a compact set  $\hat{K} \subset \mathbf{R}_{++}^{L-1}$  such that if  $\hat{z}(\hat{p}^*) = 0$ , then  $\hat{p}^* \in K$ .

Define  $\phi : \mathbf{R}_{++}^{L-1} \rightarrow \Delta^0$  by

$$\phi(\hat{p}) = \frac{(\hat{p}, 1)}{|\hat{p}_1 + \dots + \hat{p}_{L-1} + 1|}$$

Observe that  $\phi$  is one-to-one and onto, is continuous, and has continuous inverse. If  $K$  is any compact subset of  $\Delta^0$ , then  $\phi^{-1}(K)$  is the continuous image of a compact set, hence a compact subset of  $\mathbf{R}_{++}^{L-1}$ .

Hence, if the claim is false, the set of equilibrium prices is not contained in  $\phi^{-1}(K)$  for any  $K$ , so we may find a sequence  $\hat{p}_n^*$  of equilibrium prices such that

$$\phi(\hat{p}_n^*) \rightarrow p \in \Delta \setminus \Delta^0$$

By the Boundary Condition,

$$|z(\phi(\hat{p}_n^*))| \rightarrow \infty$$

but

$$\begin{aligned} z(\phi(\hat{p}_n^*)) &= z(\hat{p}_n^*, 1) \text{ (homogeneity of degree zero)} \\ &= 0 \text{ (since } \hat{z}(\hat{p}_n^*) = 0) \end{aligned}$$

a contradiction which proves the claim.

Since  $\hat{z}$  is continuous,

$$\begin{aligned} E &= \{\hat{p} \in \mathbf{R}_{++}^{L-1} : \hat{z}(\hat{p}) = 0\} \\ &= \hat{z}^{-1}(\{0\}) \\ &= \mathbf{R}_{++}^{L-1} \setminus \hat{z}^{-1}(\mathbf{R}^{L-1} \setminus \{0\}) \end{aligned}$$

is closed; since  $E$  is a closed subset of the compact set  $K$ ,  $E$  is compact. For each  $\hat{p} \in E$ , we may find  $\delta_{\hat{p}} > 0$  such that

$$E \cap B(\hat{p}, \delta_{\hat{p}}) = \{\hat{p}\}$$

The collection

$$\{B(\hat{p}, \delta_{\hat{p}}) : \hat{p} \in E\}$$

is an open cover of  $E$ , hence has a finite subcover. Since each element of this finite subcover contains exactly one element of  $E$ ,  $E$  is finite. ■