Economics 201b
Spring 2010
Problem Set 4 Solutions

1. **Robinson-Crusoe in U.S.S.R.** Consider a "Robinson-Crusoe" economy with two-goods, one consumer and one firm. Firm is labor-oriented: it maximizes the profits per unit of labor, given the wage rate w and price of potatoes p (i.e. it maximizes $\frac{\pi(p,w)}{\ell}$ where ℓ is an amount of labor).

   (a) What is the definition of competitive equilibrium in this case? Give a formal definition. Call it P ("Proletariat") equilibrium. (Yes, P equilibria of the whole world, unite!).

   Solution. One can define P equilibrium in the same way as it was defined in a familiar Robinson-Crusoe economy in Lecture 4. The only distinction is in the firm's maximization problem: rather then maximize profits firm would maximize profits, per unit of labor. So, define P equilibrium as a tuple $((p^*, w^*),\ (x_1^*, x_2^*),\ (\ell^*, q^*))$ such that

   1. $(x_1^*, x_2^*) \in \underset{x_1,x_2}{\operatorname{argmax}}\, U(x_1, x_2)$ s.th. $p^* x_2 \leq w^*(L - x_1) + \pi(p^*, w^*)$,
   2. $\ell^* \in \underset{\ell}{\operatorname{argmax}}\, \frac{p^* f(\ell) - w^* \ell}{\ell}$,
   3. $x_1^* + \ell^* = L$.

   (b) When production function $f(z)$, where z is labor input, is strictly concave what is the set of all Pareto optimal allocations?

   Solution. Again, as before, firms have no say in the determination of the Pareto optimal allocations. So, this set is just the solution to the following problem: $\underset{x_1 \in [0,L]}{\operatorname{argmax}}\, U(x_1, f(L - x_1))$ (and clearly $x_2 = f(L - x_1)$).
   Notice that under assumption of strict quasi-concavity of U, this set is just a singleton (i.e. there is a unique Pareto optimal allocation).

   (c) Continue to assume that $f(z)$ is strictly concave, under what condition on utility function does the P equilibrium exist? Give the description of the equilibrium in this case.

   Solution. First, make standard assumption $f(0) = 0$ and without any loss of generality set $p = 1$. Second, observe that under strict concavity of $f(z)$ assumption, the only solution to the firms maximization problem defined in (a), is $\ell^* = 0$. To see this, note that if $f'(\ell) = \infty$ as $\ell \to 0$ then l'Hospital rule gives $\lim_{l \to 0} \frac{f(\ell)}{\ell} = \lim_{l \to 0} \frac{f'(\ell)}{1} = +\infty$ by our assumption, so $\ell^* = 0$ is, indeed, the maximizer. Now, if $f'(\ell) \neq \infty$ as $\ell \to 0$, then FOC

implies $\ell f'(\ell) - f(\ell) = 0$. Since $\dfrac{d(\ell f'(\ell) - f(\ell))}{d\ell} = f''(\ell) < 0$, $\ell f'(\ell) - f(\ell)$ is strictly decreasing, and, thus, the only maximizer is $\ell^* = 0$. In other word, the ratio $\frac{f(\ell)}{\ell}$ is always decreasing.
Consequently, the only possibility for P equilibrium to exist is

$$(L, 0) \in \underset{x_1, x_2}{\operatorname{argmax}}\, U(x_1, x_2) \text{ s. th. } x_2 \leq 0.$$

The easiest way to achieve that is to set $U(x_1, x_2) = U(x_1)$, (i.e. Robinson does not care about consumption good, only leisure.) Aside from that possibility, non-convex utility function will generate $(L, 0)$ corner solution for range of prices.

(d) For an arbitrary production function check whether P equilibrium is Pareto efficient.

Solution. For the reasons described in (c), with a well-behaved utility function P equilibrium(-a) will not be Pareto optimal. The same holds true for strictly convex $f(z)$, with the only distinction that $\ell^* = L$ in this case.
But, with CRS technology, i.e. $f(z) = kz$, there will be at least one P equilibrium (more if utility function is not strictly quasi-concave) which is Pareto optimal. It is obtained by setting $w = \frac{1}{k}$, because with CRS any $x_1 \in [0, L]$ is a maximizer.

(e) Now, suppose that in recognition of such great management innovation, firm receives an award from *Politburo* of the economy (i.e. local social planner). Thus, the firm maximizes now $\frac{\pi(p,w)+a}{\ell}$ where $a > 0$ is the fixed award amount. If the utility function U is quasi-concave, continuous and strictly monotone, are there any conditions on the production function such that P equilibrium exist? Prove or give counterexample.

Solution. Following similar arguments as in (c) and (d), maximizing $\dfrac{\pi(p, w) + a}{\ell}$ yields $\ell^* = 0$ as unique maximizer for the case when $f(z)$ is either concave or linear. When $f(z)$ is strictly convex the unique maximizer is either $\ell^* = L$ or $\ell^* = 0$ (because $\dfrac{\pi(p, w) + a}{\ell}$ is a sum of strictly increasing function $\frac{\pi(p,w)}{\ell}$ and strictly decreasing function $\frac{a}{\ell}$, thus, it has only a unique minimizer on the interior and is maximized at the end-points.) Consequently, in general, there are no condition on $f(z)$ such that P equilibrium exists.

2. **Robinson-Crusoe: back to Berkeley.** Consider again following "Robinson-Crusoe" economies with two-goods, one consumer and one firm. For each case, compute all Pareto optimal allocations and check whether or not the Second Welfare Theorem holds. Justify your answer.

(a) $U(x_1, x_2) = \log x_1 + \log x_2,\ \omega = (24, 0),$
$Y = \{(-y_1, y_2) : y_2 \leq e^{y_1 - 1},\ y_1 \geq 0\}$

Solution. Notice that the production set is non-convex. The Pareto set is given by

$$\max_{x_1, x_2} \log x_1 + \log x_2, \text{ s.th. } x_2 \leq e^{23 - x_1}$$

The solution is given by

$$\max_{0 \leq x_1 \leq 24} \log x_1 + 23 - x_1$$

Since $\frac{\partial U}{\partial x_i} \to -\infty$ as $x_i \to 0$, the solution will always be interior. The unique maximizer is $x_1^* = 1$, thus, the Pareto optimal allocation is $(x_1^*, x_2^*) = (1, e^2 2)$. However, this allocation can't be sustained as a competitive equilibrium, because profit maximization will never yield an interior solution with non-convex production set.

(b) $U(x_1, x_2) = \log x_1 + \log x_2,\ \omega = (24, 0),$
$Y = \{(-y_1, y_2) : y_2 \leq \begin{cases} \frac{3}{4} y_1 & \text{if } 0 \leq y_1 \leq 20 \\ (y_1 - 20)^2 + 15 & \text{if } 20 < y_1 \end{cases}$

Solution. Notice that production set is linear over some range and then starts to exhibit increasing returns to scale. To find Pareto optimal allocation we need solve two maximization subproblems and then compare utility levels. For $0 \leq x_1 \leq 20$, the problem is

$$\max_{x_1, x_2} \log x_1 + \log x_2, \text{ s.th. } x_2 \leq \frac{3}{4}(24 - x_1)$$

The solution is given by

$$\max_{4 \leq x_1 \leq 24} \log x_1 + \log\left(\frac{72 - 3x_1}{4}\right).$$

Again, the solution is a unique interior maximizer $x_1 = 12$ with $U(12, 9) = \log 108$. For $x_1 > 20$, the problem is

$$\max_{x_1, x_2} \log x_1 + \log x_2, \text{ s.th. } x_2 \leq (4 - x_1)^2 + 15$$

The solution is given by

$$\max_{0 \leq x_1 < 4} \log x_1 + \log\left((4 - x_1)^2 + 15\right).$$

There is no interior solution, and, thus, we only need to check one corner $(4, 15)$. But, since $U(4, 15) = \log 60 < \log 108 = U(12, 9)$, the unique Pareto optimal allocation is $(x_1^*, x_2^*) = (12, 9)$. However, this allocation can't be sustained as a competitive equilibrium, because of the non-convexities in the production set. Notice, that it does not matter that Pareto optimal allocation lies in the region where production frontier is linear. As long as there are increasing returns to scale for $y_1 > 20$, the Second Welfare Theorem will fail to hold.

(c) $U(x_1, x_2) = 3x_1^2 + e^{x_2},\ \omega = (24, 0),$
$Y = \{(-y_1, y_2) :\ y_2 \leq \log(y_1 + 1),\ y_1 \geq 0\}$

Solution. Notice that the utility function is non-convex. The Pareto set is given by

$$\max_{x_1,x_2}\ 3x_1^2 + e^{x_2},\ \text{s.th. } x_2 \leq \log(25 - x_1)$$

The solution is given by

$$\max_{0\leq x_1\leq 24}\ 3x_1^2 + 25 - x_1$$

Notice that $x_1^* = 24$ is the maximizer, thus, the Pareto optimal allocation is $(x_1^*, x_2^*) = (24, 0)$ and it can be sustained as a competitive equilibria if $\frac{p_1}{p_2}$ is sufficiently low.
Notice that non-convexities in the utility function will, in general, lead to the failure of the Second Welfare Theorem. However, the latter only provides sufficient conditions for any Pareto optimal allocation to be a competitive equilibria with transfers and in this case the unique Pareto optimal allocation $(24, 0)$ can be sustained as a competitive equilibria. Finally, observe that firm maximization problem implies $\frac{1}{y_1+1} = w$. Consequently, setting $p^* = 1,\ w^* = 1$ generates a competitive equilibrium allocation $(24, 0)$.

3. **Quasi-equilibrium to equilibrium in economy with production.** In lecture we have shown that with strict monotonicity of preferences any price quasi-equilibrium is also a price equilibrium in pure exchange economy. Now, you need to prove that under our assumptions on preferences (continuous, convex and strongly monotone) as well as an additional assumption that $\exists y_j \in Y_j :$ $\sum_j y_j + \bar{\omega} \gg 0$ this claim is also true in the Arrow-Debreu economy with production.

Solution. Let $\{\succsim\}$ be continuous and strongly monotone, $X = \mathbb{R}_+^L$ and assume $\exists y_j \in Y_j :\ \sum_j y_j + \bar{\omega} \gg 0$. Define quasi-equilibrium with transfers (x^*, y^*, p^*, T) as an assignment of wealth levels $(w_1, ..., w_I)$ with $\sum_i w_i = p^* \cdot \bar{\omega} + \sum_i p^* \cdot y_i^*$ such that

1. $x_i \succsim_i x_i^* \implies p^* \cdot x_i \geq w_i\ \ \forall i,$
2. $\forall y_j \in Y_j :\quad p^* \cdot y_j^* \geq p^* \cdot y_j\ \ \forall j,$
3. $\sum_i x_i^* = \bar{\omega} + \sum_i y_i^*$

Step 1. Show that $p \geq 0$. Notice that $p \neq 0$ by assumption (in the proof of the Second Welfare Theorem it is a result of the application of Minkowski Separating Hyperplane Theorem). To see this, assume by contradiction that $\exists l :\ p_l < 0$. Without any loss of generality set $l = 1$, i.e. $p_1 < 0$. Now, define $x_i' = x_i^* + \left(-\frac{1}{p_1}, 0, \ldots, 0\right)$. By strict monotonicity $x_i' \succ_i x_i^*$. Now consider the cost of this bundle $p \cdot x_i' = p \cdot x_i^* + \left(p_1 \cdot (-\frac{1}{p_1})\right)$. Since $p \cdot x_i' < w_i$ this is a contradiction, thus, $p \geq 0$.

Step 2. Show $w_i > 0$ for some i. Observe that profit maximization by firms implies

$$\forall y_j \in Y_j : \quad p^* \cdot y_j^* \geq p^* \cdot y_j \ \ \forall j$$

summing it up over j

$$\sum_j p \cdot y_j^* \geq \sum_j p \cdot y_j$$

which is true if

$$\sum_j p \cdot y_j^* + p \cdot \bar{\omega} \geq \sum_j p \cdot y_j + p \cdot \bar{\omega}$$

and by our assumption $\sum_j p \cdot y_j^* + p \cdot \bar{\omega} \gg 0$. Recall that $w_i = p \cdot w_i + p \cdot \sum_j \theta_{ij} \cdot y_j^* + T_i$ and by our previous result $\sum_i w_i > 0 \implies \exists i$ such that $w_i > 0$.

Step 3. Show $p \gg 0$. Since we know that $p \geq 0$ and $p \neq 0$ by our assumption, all we need to show that $p_l = 0$ for some l leads to a contradiction. But, suppose not and there exists at one good (without loss of any generality, the first one), such that $p_1 = 0$. Consider $x_i' = x_i^* + (\varepsilon, 0, \ldots, 0)$, $\varepsilon > 0$. By strict monotonicity $x_i' \succ_i x_i^*$ and the cost of this bundle is the same $p \cdot x_i' = p \cdot x_i^* = w_i$. By continuity of preferences:

$$\text{if } x_i' \succ_i x_i^* \quad \exists \bar{\epsilon} \text{ s.th. } y_i \succ_i x_i^* \ \ \forall y_i \in B_{\bar{\epsilon}}(x_i').$$

But this implies that there exists some bundle in the budget set that costs as much as x_i^*, but is strictly preferred, which is a contradiction.

Step 4. Finally, we show that if some bundle is strictly preferred to x_i^* then it will have to cost more then x_i^* (which is equivalent to say that it is a price equilibrium). Notice that if $w_i = 0$ then both demand and quasi-demands are empty sets and we are done. If $w_i > 0$ then $p \cdot x_i^* > 0$. Now, suppose that $x_i' \succ_i x_i^*$ but $p \cdot x_i' = p \cdot x_i^* = w_i$. Now, by continuity of the preferences we have

$$\exists \bar{\epsilon} \text{ s.th. } y_i \succ_i x_i^* \ \ \forall y_i \in B_{\bar{\epsilon}}(x_i').$$

Observe that $B_{\bar{\epsilon}}(x_i') \cap B_i \neq \emptyset$, where B_i is a budget set of the consumer i. But this is a contradiction and we are done.

4. **"Tricky" Boundary Conditions.** A common misconception about the boundary condition on excess demand is to think that it says that if the price of a good goes to zero, then excess demand for *that good* goes to infinity. Although intuitively plausible, this is false even for very well-behaved preferences, since relative prices matter. Working this problem should help you avoid this misconception.

   Consider the preference relation on $\mathbb{R}^3_+$ represented by the utility function $U(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + x_2 + \frac{x_3}{1+x_3}$, and let the consumer's initial endowment be $\omega = (1, 1, 1)$.

(a) Show that U is strongly monotone, strictly concave, and continuous.

Solution. *Strong monotonicity* requires showing that $\frac{\partial U}{\partial x_l} > 0$ for each good l. We have

$$\begin{aligned}
\frac{\partial U}{\partial x_1} &= \frac{1}{2}x_1^{-1/2} > 0 \quad \forall x_1 \in \mathbb{R}_{++} \\
\frac{\partial U}{\partial x_2} &= \frac{1}{2}x_2^{-1/2} + 1 > 0 \quad \forall x_2 \in \mathbb{R}_{++} \\
\frac{\partial U}{\partial x_3} &= \frac{1}{(1+x_3)^2} > 0 \quad \forall x_3 \in \mathbb{R}_{++}
\end{aligned}$$

Strict concavity requires that the Hessian matrix of second derivatives be negative definite. We have:

$$D^2U(x_1, x_2, x_3) = \begin{bmatrix} \frac{-1}{4x_1^{3/2}} & 0 & 0 \\ 0 & \frac{-1}{4x_2^{3/2}} & 0 \\ 0 & 0 & \frac{-2}{(1+x_3)^2} \end{bmatrix}$$

which is negative definite $\forall x \in \mathbb{R}^3_{++}$. *Continuity.* Note that $\sqrt{x_1}$, $\sqrt{x_2}$, x_2, and $\frac{x_3}{1+x_3}$ are all continuous functions, and that the sum of continuous functions is continuous, so U is continuous.

(b) If $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ and $x_3 > 0$, show that $U(x_1, x_2 + x_3, 0) > U(x_1, x_2, x_3)$

Solution. $U(x_1, x_2 + x_3, 0) = \sqrt{x_1} + \sqrt{x_2 + x_3} + x_2 + x_3$. Note that $x_3 > \frac{x_3}{1+x_3}$ $\forall x_3 \in \mathbb{R}_+$, and $\sqrt{x_2 + x_3} > \sqrt{x_2}$ $\forall x_2,\ x_3 \in \mathbb{R}_+$. Thus if $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ and $x_3 > 0$, then $U(x_1, x_2 + x_3, 0) > U(x_1, x_2, x_3)$.

(c) If $p = (p_1, p_2, p_3) >> 0$ and $p_2 = p_3$, show that $x_3(p) = 0$.

Solution. Proceed by contradiction and suppose $x_3(p) > 0$. From part (b) above, we know that $U\big(x_1(p), x_2(p)+x_3(p), 0\big) > U\big(x_1(p), x_2(p), x_3(p)\big)$. Also, because $p_2 = p_3$, we have

$$p \cdot \big(x_1(p), x_2(p) + x_3(p), 0\big) = p \cdot \big(x_1(p), x_2(p), x_3(p)\big).$$

These two facts contradict $\big(x_1(p), x_2(p), x_3(p)\big)$ being demanded at price p, so we must have $x_3(p) = 0$.

(d) For each n, let $p^n = (1 - \frac{2}{n}, \frac{1}{n}, \frac{1}{n})$. Show that $x_3(p^n) = 0$ for each n (and thus that demand for x_3 remains bounded even though $p_3^n \to 0$).

Solution. For this sequence, we have $p^n \gg 0$ and $p_2^n = p_3^n$ $\forall n$. Thus, by part (c), $x_3(p^n) = 0$ for each n.

(e) Show that $\lim_{n\to\infty} x_2(p^n) = \infty$.

Solution. From what we learned about preferences in part (a) and the proof of question 4, we know that the boundary condition on excess demand must hold. That is, $|z(p^n)| \to \infty$. Demand for at least one of the goods must be blowing up. And since $x_3(p^n) = 0\ \forall n$, it must be either good 1 or good 2 that blows up.

If $x_1(p^n) \to \infty$, Then we have that $p^n \cdot x(p^n) \to \infty$ which violates Walras' Law (which must hold given strong monotonicity). Thus, we must have $\lim_{n\to\infty} x_2(p^n) = \infty$.

5. **Importance of Assumptions.** Consider a two good economy, and illustrate graphically four examples of functions $z : \Delta^o \to \mathbb{R}^2$ which demonstrate that if any one of the conditions: continuity; Walras' Law; boundary condition; bounded below fails, then there may not be a solution to $z(p) = 0$. That is, each function you draw should violate only one of the four conditions, and have the property that $\nexists p$ s.t. $z(p) = 0$.

(a) Continuity

Solution. Consider following $z(p)$

$$z(p) = \begin{cases} \begin{bmatrix} \frac{\alpha p\omega}{p_1} - \omega_1 \\ \frac{(1-a)p\omega}{p_2} - \omega_2 \end{bmatrix}, & \text{if } p \neq p^* \\ \begin{bmatrix} \frac{k}{p_1^*} \\ -\frac{k}{p_2^*} \end{bmatrix} \neq 0 \quad \forall k > 0, & \text{if } p = p^* \end{cases}$$

Notice that $z(p)$ as defined above for $p \neq p^*$ represent excess demand for a Cobb-Douglas "representative consumer," so that there is a unique $p^* \in \Delta^o$ such that $z(p) \neq 0$ (for Cobb-Douglas preferences AD curves are always slope downwards). So, if one redefines excess demand in just one point, $p = p^*$, the result is discontinuous excess demand function $z(p)$ that satisfies all other properties and, clearly, no Walrasian equilibrium in that economy.

Note that $z(p)$ satisfies Walras' Law

$$p \cdot z(p) = \begin{cases} \frac{\alpha \cdot p \cdot \omega}{p_1} p_1 - p_1 \cdot \omega_1 + \frac{(1-a) \cdot p \cdot \omega}{p_2} p_2 - p_2 \cdot \omega_2 & \text{if } p \neq p^* \\ = 0 & \text{if } p = p^* \end{cases}$$

which implies

$$p \cdot z(p) = \begin{cases} p \cdot \omega - p_1 \cdot \omega_1 - p_2 \cdot \omega_2 = 0 & \text{if } p \neq p^* \\ = 0 & \text{if } p = p^* \end{cases}$$

is bounded below

$$z(p) \geq \begin{cases} -\max(\omega_1, \omega_2) & \text{if } p \neq p^* \\ -\max\left\{\frac{k}{p_1^*}, \frac{k}{p_2^*}\right\} & \text{if } p = p^* \end{cases} \Longrightarrow$$

$$z(p) \geq -\max\left\{\omega_1, \omega_2, \frac{k}{p_1^*}, \frac{k}{p_2^*}\right\}$$

and satisfies boundary conditions: if $p_l \to \infty$ $z_l(p) \to \infty$ since

$$\lim_{p_1 \to \infty} \frac{\alpha \cdot p \cdot \omega}{p_1} - \omega_1 \to \infty$$
$$\lim_{p_2 \to \infty} \frac{(1-\alpha) \cdot p \cdot \omega}{p_2} - \omega_2 \to \infty$$

(and we do not need to check for boundary conditions when $p = p^*$).
See figure 5(a).

(b) Walras' Law

Solution. Again, consider the excess demand function $z(p)$ for a Cobb-Douglas "representative consumer" and translate the curve so that it does not pass through the origin

$$z(p) = \begin{bmatrix} \frac{\alpha \cdot p \cdot \omega}{p_1} - \omega_1 \\ \frac{(1-a) \cdot p \cdot \omega}{p_2} - \omega_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Note that $z(p)$ as defined above is: continuous as a sum of two continuous functions; bounded below and satisfies boundary conditions (see the arguments given above). Also, one can check that $p \cdot z(p) = -p_1 c_1 - p_2 c_2 \neq 0$ whenever $p_1 c_1 \neq p_2 c_2$.
See figure 5(b).

(c) boundedness below ($\exists x \in \mathbb{R}$ s.t. $z(p) \geq x \ \forall p \in \Delta^o$),

Solution. ($\exists\ x \in \mathbb{R}$ s.th. $z(p) \geq x \ \forall p \in \Delta^o$). Consider

$$z(p) = \begin{bmatrix} \frac{1}{p_1} \\ -\frac{1}{p_2} \end{bmatrix}$$

Note that it is clearly a continuous function because $\frac{1}{p_l}$ is a continuous function for $\forall p \in \Delta^o$, satisifies Walras Law and boundary conditions, but is unbounded below because.

$$\nexists M > 0 \text{ s. th. } z_l(p) > -M \ \ \forall l, \ \forall p.$$

See figure 5(c).

(d) boundary condition (if $p^n \to p \in \Delta \backslash \Delta^o$, then $|z_l(p^n)| \to \infty$)

Solution. Consider following $z(p)$

$$z(p) = \frac{p \cdot a}{k} b - \frac{p \cdot b}{k} a$$

where $a, b \in \mathbb{R}^2_{++}$, $k \in \mathbb{R}_{++}$ and $p \in \Delta$. It is easy to check that continuity, Walras' Law and bounded below conditions are satisfied but boundary conditions not.

See figure 5(d).

6. **Continuity of correspondences.** Let $\psi : \Delta \to 2^{\Delta}$ is a correspondence. Show that ψ is uhc if it has a closed graph. Demonstrate graphically an example of correspondence $\psi : X \to 2^X$ such that ψ has a closed graph but ψ is not uhc.

Solution. Let $\psi : \Delta \to 2^{\Delta}$ has a closed graph. Then by the definition M.H.3 in MWG we only need to show that the image of compact set under ψ is bounded. So, let $K \subset \Delta$ be compact. Since $\psi(K) \subset \Delta$, $\psi(K)$ is bounded. Thus, ψ is uhc.

Please see figure 6 for the example of correspondence that has a closed graph but not uhc.

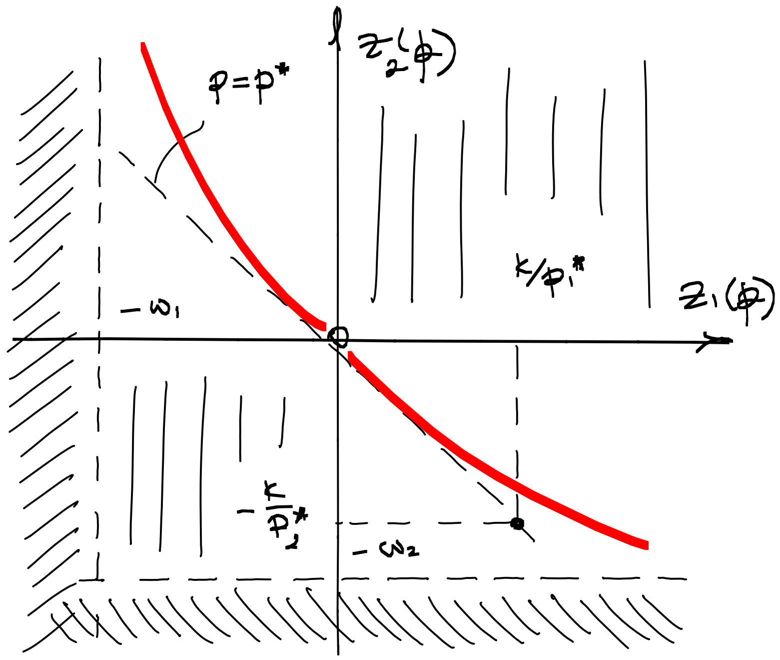

Figure 5(a). Continuity

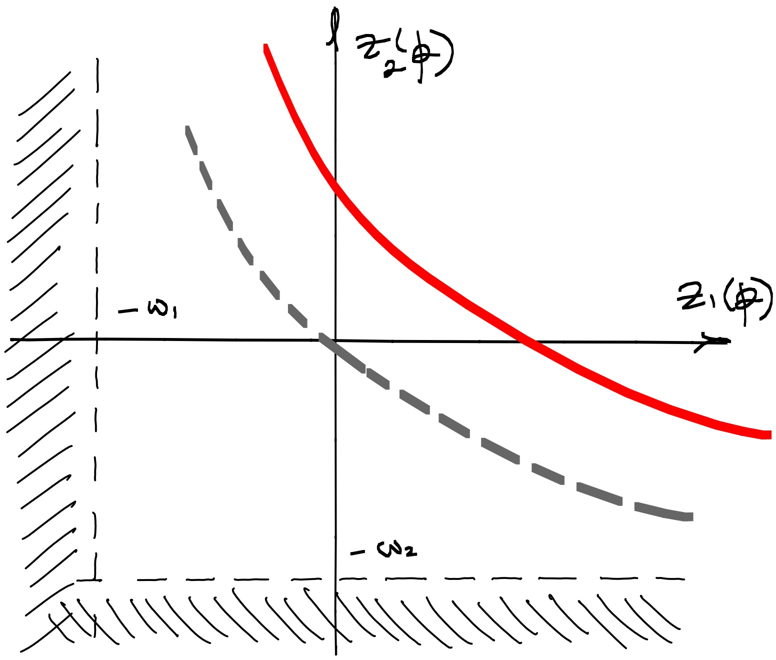

Figure 5(b). Walras' Law

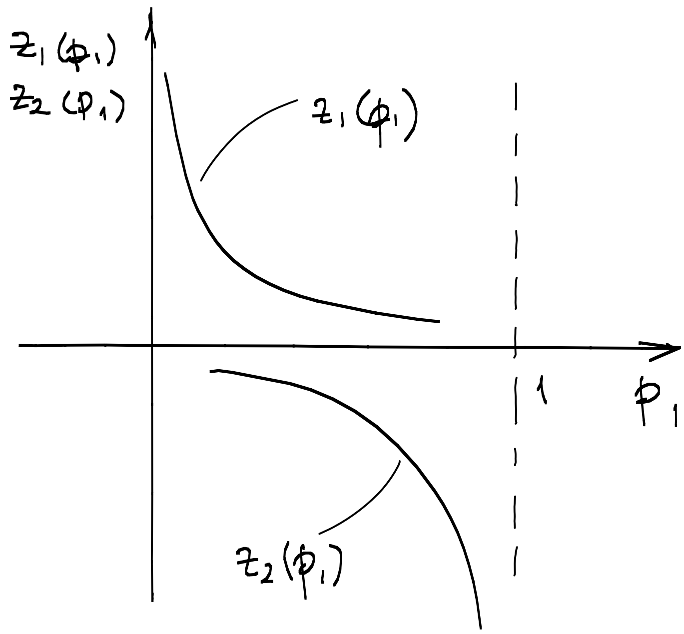

Figure 5(c). Bounded below

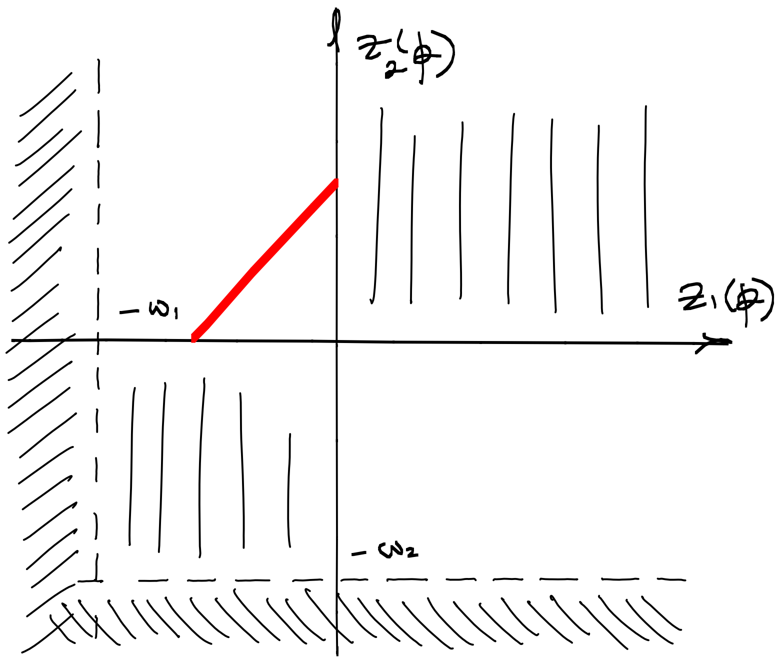

Figure 5(d) Boundary Condition.

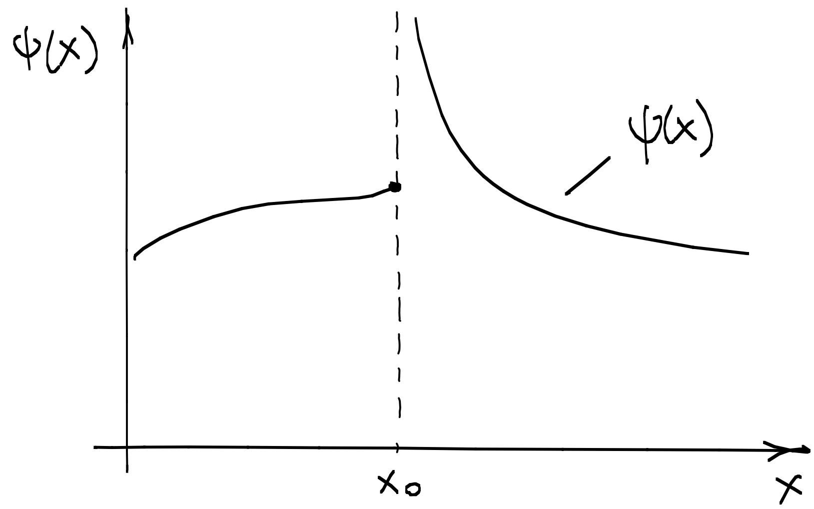

Figure 6. Correspondence that has a closed graph but not uhc.

(Of course, this is not the only example. But, be sure to understand why it works. In particular, be sure to understand why $\phi(x)$ has a closed graph.)