Economics 201b
Spring 2010
Problem Set 5
Due Thursday April 22

You will need to know the following trigonometric identities:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\cos(\theta) = \sin(\theta + \frac{\pi}{2})$$

$$\cos(\theta + \pi) = -\cos(\theta) \qquad \sin(\theta + \pi) = -\sin(\theta)$$

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

1. Give a description of an exchange economy that satisfies all the conditions of the one in Theorem 1 Lecture 6 except the condition $\bar{\omega} \gg 0$ (which is implicitly assumed in the Theorem), such that the Second Welfare Theorem now no longer holds - i.e. there is now a Pareto Optimal allocation that can't be supported as a Walrasian Equilibrium with Transfers.

2. Assume the function $F : \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{LI}_{+} \longrightarrow \mathbb{R}^{L-1}$ from Definition 1 Lecture 10 is continuous. Prove Proposition 2. (Recall, $f : U \to V$ is continuous if for every closed set $D \subset V$, $f^{-1}(D)$ is closed.)

3. Given a price $p = (p_1, p_2) \in \Delta^o$, we can define $\theta(p)$ (measured in radians) to be the angle between p and the horizontal axis. Suppose we have a function $z : \Delta^o \to \mathbb{R}^2$

$$z(p) = \left(f(\theta(p))\cos(6\theta(p))\cos(\theta(p) + \frac{\pi}{2}), f(\theta(p))\cos(6\theta(p))\sin(\theta(p) + \frac{\pi}{2}) \right)$$

where f is a positive differentiable function defined on $(0, \frac{\pi}{2})$ such that

$$\lim_{\theta \downarrow 0} f(\theta) = \lim_{\theta \uparrow \frac{\pi}{2}} f(\theta) = \infty$$

and

$$f(\theta) = \begin{cases} \leq M\frac{1}{\theta} & \theta < \epsilon \\ 1 & \theta \in [\epsilon, \frac{\pi}{2} - \epsilon] \\ \leq M\frac{1}{\frac{\pi}{2} - \theta} & \theta > \frac{\pi}{2} - \epsilon \end{cases} \tag{1}$$

for some constant $M > 0$.

(a) Draw a picture of $z(p)$.

(b) Express p_1 and p_2 as functions of $\theta(p)$.

(c) Prove $z(p) \cdot p = 0$.

(d) Prove $z(p)$ satisfies the conditions of D-G-K-N Lemma. (Hint: $z(p)$ can be rewritten as $Z(\theta(p))$. To prove that $z(p)$ is bounded below on Δ^o, it suffices to show $Z(\theta)$ is bounded below on $(0, \frac{\pi}{2})$ (notice I wrote $Z(\theta)$ *not* $Z(\theta(p))$). First **write down** $Z(\theta)$. Then to prove $Z(\theta)$ is bounded below, break $(0, \frac{\pi}{2})$ into three pieces: the compact interval $[\epsilon, \frac{\pi}{2} - \epsilon]$ and the two pieces $(0, \epsilon)$ and $(\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$. Then consider each piece separately. To show $Z(\theta)$ is bounded on $(0, \epsilon)$, it suffices to show $\lim_{\theta \to 0} Z(\theta)$ is bounded below. A similar method may be applied to the interval $(\frac{\pi}{2} - \epsilon, \frac{\pi}{2})$.)

(e) Suppose $z(p)$ was the excess demand function of some economy. What are the equilibrium prices $p_i^* = (p_{1i}^*, p_{2i}^*) \in \Delta^o$, $i = 1, 2, \ldots N$ (where N is the number of equilibrium prices you find)? Then re-express each such price p_i^* in the normalized form $(\hat{p}_i^*, 1)$. (Hint: Equilibrium prices correspond to the solutions of $z(p) = 0$. Once again, it is easier to first look for the solutions of $Z(\theta) = 0$. Then for each such solution θ_i^*, express the corresponding price p_i^*.)

(f) Recall the function $\phi(\hat{p}) \to \Delta^o$. Explain what the sign (-1, 1, or 0) of

$$\frac{d}{d\hat{p}} \theta(\phi(\hat{p}))$$

is.

(g) Write out $\hat{z}(\hat{p})$ and calculate

$$\text{index}(\hat{p}_i^*) = -\text{sgn} \frac{d}{d\hat{p}} \hat{z}(\hat{p}) \bigg|_{\hat{p} = \hat{p}_i^*} \qquad \text{for all} \quad i$$

Is the economy regular? Notice that the Index Theorem is indeed true.

(h) Suppose we change $f(\theta)$ slightly so that f is still smooth but now $f > 0$ except at a single point $x \in (0, \frac{\pi}{2})$ where $f(x) = 0$ - rename this altered function $f_x(\cdot)$. Consider the new excess demand function $z_x(p)$ where the old f is replaced with the new f_x (convince yourself $z_x(p)$ still satisfies the conditions of D-G-K-N). The corresponding economy is no longer regular. The summation in the Index Theorem can now take the values 0, 1, and 2 depending on the value x. Which values of x correspond to which values of the summation?