Economics 250a
Hausman's (1981) linear demand system – an application of integrability.

Reference: Jerry Hausman, "Exact Consumer's Surplus and Deadweight Loss" AER 71 (Sept 1981), 662-676.

Hausman considers a linear demand model for a consumer who can choose between 2-goods:

$$x = x(p, y) = \alpha p + \delta y + \gamma + \varepsilon \tag{1}$$

where x is the demand for the commodity of interest, p is the price of the commodity, y represents income, γ is a constant (that can potentially vary across people depending on demographics) and ε is a pure error term (not part of preferences; possibly measurement error in x). The integrability theorem says that if this demand function is preference-generated then it is associated with an expenditure function $e(p, u)$ that solves the differential equation:

$$\frac{\partial e(p, u)}{\partial p} = \alpha p + \delta e(p, u) + \gamma. \tag{2}$$

Note we ignore the ε. That is the sense in which it is interpreted as "noise", or something other than preferences. Hausman shows that the solution is in the class

$$e(p, u) = c \exp(\delta p) - \frac{1}{\delta}(\alpha p + \frac{\alpha}{\delta} + \gamma)$$

where c is a constant of integration that can depend on u. Since utility can be arbitrarily transformed, Hausman simply sets $c = u$. So the proposed solution is

$$e(p, u) = u \exp(\delta p) - \frac{1}{\delta}(\alpha p + \frac{\alpha}{\delta} + \gamma) \tag{3}$$

Checking this potential solution, note that (3) implies

$$\frac{\partial e(p, u)}{\partial p} = \delta u \exp(\delta p) - \frac{\alpha}{\delta} \tag{4}$$

and plugging into (2) you can see it works.

Is this a "proper" expenditure function? It will be if

$$\frac{\partial^2 e(p, u)}{\partial p^2} \leq 0$$

which will require

$$\delta^2 u \exp(\delta p) \leq 0.$$

Now from (3)

$$u \exp(\delta p) = e(p, u) + \frac{1}{\delta}(\alpha p + \frac{\alpha}{\delta} + \gamma) \tag{5}$$

$$\Rightarrow \delta^2 u \exp(\delta p) = \delta(\delta y + \alpha p + \frac{\alpha}{\delta} + \gamma) \qquad (substituting\ y = e)$$

$$= \delta(x + \frac{\alpha}{\delta}) = \alpha + \delta x \tag{6}$$

Applying Slutsky's equation, notice that if we have the linear demand system (1) then

$$\begin{aligned}\frac{\partial x^c}{\partial p} &= \frac{\partial x}{\partial p} + \frac{\partial x}{\partial y}x \\ &= \alpha + \delta x\end{aligned}$$

So if the "Slutsky condition" $\frac{\partial x^c}{\partial p} \leq 0$ the expenditure function will be concave.