Economics 202A
Suggested Solutions to Problem Set 2

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1 Romer, 1.11. Embodied technological progress

(a) We modify first the Solow model so that $Y(t) = [A(t)K(t)]^{\alpha} L(t)^{1-\alpha}$ and the growth rate of A is now μ. We now have to establish the convergence to the balanced growth path and find the growth rates of K and Y on the balanced growth path.

Method 1: Using the hint, we define $k = \frac{K}{A^{\phi}L}, y = \frac{Y}{A^{\phi}L}$, where $\phi = \frac{\alpha}{1-\alpha}$. Then

$$y = \frac{A^{\alpha}K^{\alpha}L^{1-\alpha}}{A^{\frac{\alpha}{1-\alpha}}L} = \frac{K^{\alpha}}{A^{\frac{\alpha^2}{1-\alpha}}L^{\alpha}} = \left[\frac{K}{A^{\frac{\alpha}{1-\alpha}}L}\right]^{\alpha} = k^{\alpha}$$

We know that as before $\dot{K} = sY - \delta K$. Using this fact and taking logs and derivatives of the definition of k with respect to time, we get

$$\begin{aligned}
\frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \phi\frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \frac{sA^{\alpha}K^{\alpha}L^{1-\alpha}}{K} - \delta - \phi\mu - n = \\
&= \frac{sA^{\alpha}L^{1-\alpha}}{K^{1-\alpha}} - (\delta - \phi\mu - n) = \frac{s}{k^{1-\alpha}} - (n + \phi\mu + \delta), \\
\dot{k} &= sk^{\alpha} - (n + \phi\mu + \delta)k.
\end{aligned}$$

So, on the balanced growth path $sk^{\alpha} = (n + \phi\mu + \delta)$. Now we show using the diagram that the economy actually converges to the balanced growth path (see Figure 1). Our function satisfies all the properties to insure that. If $k < k*$, then $\dot{k} > 0$ and k converges to $k*$. If $k > k*$, then $\dot{k} < 0$ and k also converges to $k*$.

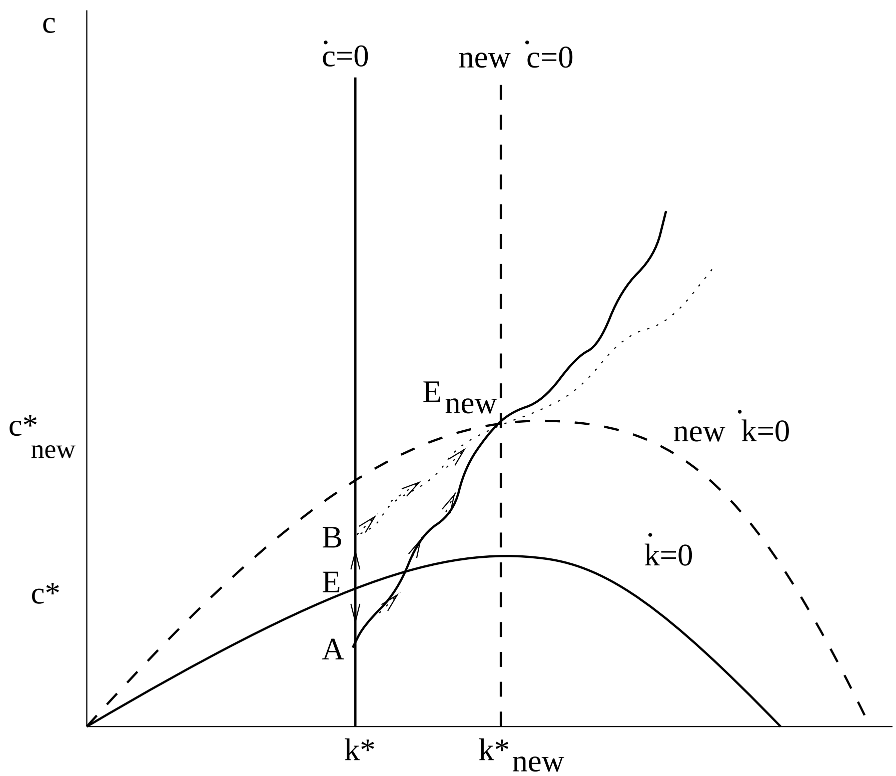

Figure 1: Convergence to the BGP.

From the definitions of k and y we know that when k is constant, on the balanced growth path, y is constant and K and Y grow at the same rate of $\mu\phi + n$.

Method 2: we can calculate growth rates without using k.

$$
\begin{aligned}
\frac{\dot{Y}}{Y} &= \alpha\frac{\dot{A}}{A} + \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}, \\
g_Y &= \alpha\mu + \alpha g_K + (1-\alpha)n.
\end{aligned}
$$

Then, if the balanced growth path exists, the growth rates g_Y, g_K should be constant. Wee also know form the equation of motion of capital that

$$\frac{\dot{K}}{K} = g_K = s\frac{Y}{K} - \delta.$$

For g_K to be constant the ratio $\frac{Y}{K}$ must be constant, which implies that $g_Y = g_K$ on the balanced growth path. So we can derive that

$$g_Y = g_K = \frac{\alpha}{1-\alpha}\mu + n = \phi\mu + n.$$

(b) Now $Y(t) = J(t)^\alpha L(t)^{1-\alpha}$ and $\dot{J(t)} = sA(T)Y(t) - \delta J(t)$. We will use method 1 to show the convergence to the balanced growth path.

$$\begin{aligned}
\bar{J} &= \frac{J}{A},\ j = \frac{\bar{J}}{A^\phi L} = \frac{J}{A^{\frac{1}{1-\alpha}}L} \\
y &= \frac{J^\alpha L^{1-\alpha}}{A^{\frac{\alpha}{1-\alpha}}L} = j^\alpha,
\end{aligned}$$

we see that our production function satisfies all the important assumptions. Taking logs and derivatives of the definition of j we find that

$$\begin{aligned}
\frac{\dot{j}}{j} &= \frac{\dot{J}}{J} - \frac{1}{1-\alpha}\frac{\dot{A}}{A} - \frac{\dot{L}}{L} = \\
&= \frac{s}{j^{1-\alpha}} - \delta - \frac{1}{1-\alpha}\mu - n, \\
\dot{j} &= sj^\alpha - (\frac{1}{1-\alpha}\mu + n + \delta)j.
\end{aligned}$$

By analogy with part (a) we can conclude that the economy converges to its balanced growth path determined by the following condition:

$$sj^\alpha = (\frac{1}{1-\alpha}\mu + n + \delta)j = (1+\phi)\mu + n + \delta)j.$$

On the balanced growth path, j and y are constant and thus J grows at the constant rate of $(1+\phi)\mu + n$ and Y grows at the constant rate $\phi\mu + n$.

(c) We can determine from the result in (b) that

$$j^* = \left[\frac{s}{(1+\phi)\mu + n + \delta}\right]^{\frac{1}{1-\alpha}}$$

and that

$$y^* = j^{*\alpha} = \left[\frac{s}{(1+\phi)\mu + n + \delta}\right]^{\frac{\alpha}{1-\alpha}},$$

which is the same as in Solow model except for the technological growth rate term. Thus, as in Solow model, the elasticity of $y*$ with respect to s will be equal to $\frac{\alpha}{1-\alpha}$.

(d) As we did in section, consider the first order approximation of $\dot{y}$ around the BGP $y = y*$ noting that since $y = j^\alpha$, $\dot{y} = \alpha j^{\alpha-1}\dot{j}$:

$$\begin{aligned}
\dot{y} &\cong \frac{\partial \dot{y}}{\partial y}|_{y=y^*}(y - y^*); \\
\dot{y} &= \alpha j^{\alpha-1}[sj^\alpha - (n + \delta + \mu(1+\phi)j)] = \\
&= \alpha s j^{2\alpha-1} - \alpha j^\alpha(n + \delta + \mu(1+\phi)) = \\
&= \alpha s y^{\frac{2\alpha-1}{\alpha}} - \alpha y(n + \delta + \mu(1+\phi)), \\
\frac{\partial \dot{y}}{\partial y}|_{y=y^*}(y - y^*) &= (2\alpha - 1)s y^{*\frac{\alpha-1}{\alpha}} - \alpha(n + \delta + \mu(1+\phi)) = \\
&= (2\alpha - 1)(n + \delta + \mu(1+\phi)) - \alpha(n + \delta + \mu(1+\phi)) = \\
&= -(1-\alpha)(n + \delta + \mu(1+\phi)), \\
\dot{y} &\cong \overbrace{-(1-\alpha)(n + \delta + \mu(1+\phi))}^{\lambda}(y - y^*).
\end{aligned}$$

If we assume that $\mu = g$, then this speed of convergence λ is higher than in Solow model, because $\phi = \frac{\alpha}{1-\alpha}$ is positive.

2 Natural resources

In this version of Solow model $Y = F(Z, R)$ is CRS.

$$\begin{aligned}\varepsilon &= \frac{\partial \ln(\frac{F_Z}{F_R})}{\partial \ln(\frac{Z}{R})} \\ \sigma = -\frac{1}{\varepsilon} &\implies \varepsilon = -\frac{1}{\sigma}.\end{aligned}$$

(a) The factor shares are defined as usual:

$$\alpha_R = \frac{F_R R}{F_Z Z + F_R R},\ \alpha_Z = \frac{F_Z Z}{F_Z Z + F_R R}.$$

Then $\frac{\alpha_Z}{\alpha_R} = \frac{F_Z}{F_R}\frac{Z}{R}$ and $\ln\frac{\alpha_Z}{\alpha_R} = \ln\frac{F_Z}{F_R} + \ln\frac{Z}{R}$. The elasticity is then

$$\begin{aligned}\varepsilon_\alpha = \frac{\partial \ln \frac{\alpha_Z}{\alpha_R}}{\partial \ln \frac{Z}{R}} &= \frac{\partial \ln \frac{F_Z}{F_R}}{\partial \ln \frac{Z}{R}} + 1 = \varepsilon + 1 = 1 - \frac{1}{\sigma} \\ \frac{\partial \ln \frac{\alpha_Z}{\alpha_R}}{\partial \ln \frac{Z}{R}} &= 1 - \frac{1}{\sigma}\end{aligned}$$

(b) We are given that $(\ln \dot{\frac{Z}{R}}) = 0.02$, therefore we can write that $\frac{Z}{R}(50) = e^{0.02*50}\frac{Z}{R}(0)$, which is equivalent to $\frac{\frac{Z}{R}(50)}{\frac{Z}{R}(0)} = e^1$, or taking logs, $\Delta \ln \frac{Z}{R} = 1$.

We can rewrite the result in (a) in discrete terms now:

$$\Delta \ln \frac{\alpha_Z}{\alpha_R} = (1 - \frac{1}{\sigma})\Delta \ln \frac{Z}{R}.$$

We are given that $\sigma = 0.8$, therefore $1 - \frac{1}{\sigma} = -0.25$. Also, because $\alpha_R = 0.15$ we can calculate $\ln \frac{\alpha_Z}{\alpha_R}(0) = \ln \frac{0.85}{0.15} = \ln 5.6 = 1.72$. Substituting, we get that

$$\Delta \ln \frac{\alpha_Z}{\alpha_R} = -0.25 \Longrightarrow \ln \frac{\alpha_Z}{\alpha_R}(50) = 1.72 - 0.25 = 1.47.$$

Thus, $\frac{\alpha_Z}{\alpha_R} = 4.36$ and, solving for α_R, we get that $\alpha_R \cong 0.3 = 30\%$.

(c) (i) $\sigma = 0.6$, therefore $1 - \frac{1}{\sigma} = -0.66$

$$\ln \frac{\alpha_Z}{\alpha_R}(50) = 1.72 - 0.66 = 1.06 \Longrightarrow \alpha_R \cong 0.53 = 53\%.$$

(ii) $\sigma = 1.25$, therefore $1 - \frac{1}{\sigma} = 0.2$

$$\ln \frac{\alpha_Z}{\alpha_R}(50) = 1.72 + 0.2 = 1.92 \Longrightarrow \alpha_R \cong 0.17 = 17\%.$$

As the elasticity of substitution between factors increases, which means that scarce factor is easier to substitute for, the share of this factor grows slower and slower. This is the result we would expect to see.

3 Romer, 2.3. Log utility

We have to find the path of consumption per worker in the Ramsey model given that $u(C) = \ln C$.

The Household is maximizing (2.1) in the book subject to (2.5) in the book. We will normalize the number of households to 1 for simplicity. We also know that the budget constraint is binding for the maximizing household.

$$\max \int_0^\infty \mathrm{e}^{-\rho t} \ln C(t) L(t) \mathrm{d}t$$

s.t.

$$\int_0^\infty \mathrm{e}^{-R(t)} C(t) L(t) \mathrm{d}t = \overbrace{K(0) + \int_0^\infty \mathrm{e}^{-R(t)} A(t) w(t) L(t) \mathrm{d}t}^{W},$$

Where W is wealth plus the present value of the labor income. Then we have to find $C(W, R(t), \rho$, initial conditions). As this problem is the special case of the one considered in class (with $\theta = 1$), we can use Lagrangian to solve it. I will use Hamiltonian to show you the method. But first we have to rewrite the life-time budget constraint as a period budget constraint. We can do this simply by differentiating.

First, rewrite budget constraint as of time s:

$$\int_s^\infty \mathrm{e}^{-(R(t)-R(s))} C(t) L(t) \mathrm{d}t = K(s) + \int_s^\infty \mathrm{e}^{-(R(t)-R(s))} A(t) w(t) L(t) \mathrm{d}t.$$

Now define $k = \frac{K}{AL}, c = \frac{C}{A}$ and divide through by A(s)L(s):

$$\int_s^\infty \mathrm{e}^{-(R(t)-R(s))+(n+g)(t-s)} c(t) \mathrm{d}t = k(s) + \int_s^\infty \mathrm{e}^{-(R(t)-R(s))+(n+g)(t-s)} w(t) \mathrm{d}t.$$

We will differentiate this with respect to s, but notice first that

$$\frac{\partial(R(t)-R(s))}{\partial s} = -r(s),$$

from the definition of $R(t)$, and thus the result of the differentiation is (recall Leibniz Rule):

$$-c(s) + \int_s^\infty (r(s)-n-g)\mathrm{e}^{-(R(t)-R(s))+(n+g)(t-s)}c(t)\mathrm{d}t =$$
$$\dot{k(s)} - w(s) + \int_s^\infty (r(s)-n-g)\mathrm{e}^{-(R(t)-R(s))+(n+g)(t-s)}w(t)\mathrm{d}t.$$

We can now take the constants out of the integral (note that $r(s)$ is constant with respect to t) and replace the remaining integrals with $k(s)$, because that is what they are. We get

$$\dot{k(s)} = w(s) - c(s) + r(s)k(s) - (n+g)k(s),$$

which is the period constraint we are going to use in our maximization. But before writing down the Hamiltonian we have to rewrite the objective function in terms of "per unit of effective labor". Note that $\ln C(t) = \ln c(t) + \ln A(t) = \ln c(t) + \ln A(0)\mathrm{e}^{gt} = \ln c(t) + \ln A(0) + gt$. I will also divide the maximand by the constant L(0).

$$\max_{c(t)} \int_0^\infty \mathrm{e}^{-(\rho-n)t}(\ln c(t) + \ln A(0) + gt)\mathrm{d}t$$

s.t.

$$\dot{k(t)} = w(t) - c(t) + r(t)k(t) - (n+g)k(t).$$

The present value Hamiltonian is then

$$H = \mathrm{e}^{-(\rho-n)t}[\ln c(t) + \ln A(0) + gt + \lambda(w(t) - c(t) + r(t)k(t) - (n+g)k(t))]$$

and the first order conditions are

$$\begin{aligned} \frac{\partial H}{\partial c} &= 0 \Longrightarrow \frac{1}{c(t)} = \lambda, \text{ for all } t, \\ -\frac{\partial H}{\partial k} &= \dot{\lambda} - (\rho-n)\lambda = -\lambda(r(t)-n-g) \Longrightarrow \dot{\lambda} = \lambda(\rho+g-r(t)). \end{aligned}$$

If we differentiate the first one and plug it in the second one, we get

$$\frac{\dot{c(t)}}{c(t)} = r(t) - \rho - g$$

which implies that

$$c(t) = c(0)\mathrm{e}^{R(t)-(\rho+g)t}$$

and gives us the path of consumption per unit of effective labor. But we are interested in the consumption per person:

$$C(t) = c(t)A(t) = A(t)c(0)\mathrm{e}^{R(t)-(\rho+g)t} = A(0)c(0)\mathrm{e}^{R(t)-(\rho+g-g)t} = C(0)\mathrm{e}^{R(t)-\rho t}.$$

The only thing we are left to determine is $C(0)$. We will use the life-time budget constraint to find it. Rewrite the budget constraint substituting the expression for $C(t)$:

$$\int_0^\infty \mathrm{e}^{-R(t)+R(t)-(\rho-n)t}C(0)L(0)\mathrm{d}t = W.$$

Note that the solution exists for $\rho > n$. Taking this integral we find

$$C(0)L(0)\frac{1}{\rho - n} = W \Longrightarrow C(0) = \frac{W}{L(0)}(\rho - n).$$

Finally, we can substitute it into our equation for $C(t)$ to get

$$C(t) = \frac{W}{L(0)}(\rho - n)\mathrm{e}^{R(t)-\rho t}.$$

We see that consumption increases if the interest rate is higher than the intertemporal discount rate and its level is larger the larger the lifetime wealth and the smaller is the initial size of the family.

4 Romer, 2.6. Playing with the phase diagram

We will use our equations for the dynamics of k and c:

$$\begin{aligned} \dot{c} &= c\,\frac{f'(k) - \rho - \theta g}{\theta} \\ \dot{k} &= f(k) - (n+g)k - c. \end{aligned}$$

(a) When θ goes up, the $\dot{k}$ equation is unaffected and thus the locus $\dot{k} = 0$ is unaffected. In the equation for $\dot{c}$ the increase in θ will decrease the RHS, so for the $\dot{c}$ to be zero, $f'(k)$ must increase for every c which means that k must decrease for every c. Therefore the only change to the graph is the shift of $\dot{c} = 0$ locus to the left (see Figure 2). The new steady state values of k and c are lower then initial.

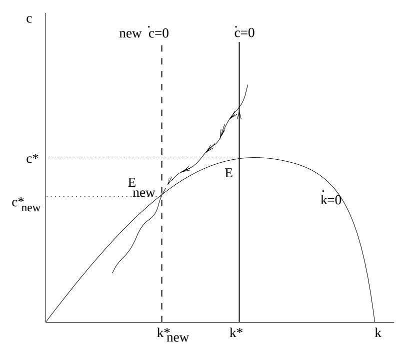

Figure 2: Θ goes up

(b) When the production function shifts downward, the $\dot{k} = 0$ locus shifts down, because for k to be constant, c must go down for every k. Also, for a given k, $f'(k)$ is now lower and therefore the $\dot{c} = 0$ locus shifts to the left (see Figure 3). As a result, the new steady state values of k and c are lower then initial.

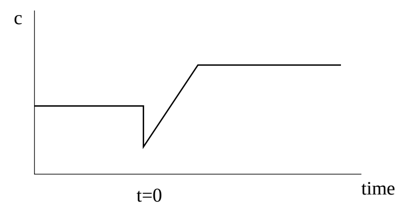

Figure 3: Production function shifts down OR Rate of depreciation is positive

(c) Now the depreciation rate is positive. This increases the break-even investment and the equations of motion change. In the equation of motion for c, δ is not part of β, but it enters the $\dot{k}$ equation in the same manner as $(n+g)$. The new equation of motion will be:

$$\begin{aligned}\dot{c} &= c\,\frac{f'(k) - \rho - \theta g - \delta}{\theta} \\ \dot{k} &= f(k) - (n+g+\delta)k - c.\end{aligned}$$

To keep $\dot{k} = 0$ now, c must fall for every k, therefore $\dot{k} = 0$ locus shifts down. To keep $\dot{c} = 0$, $f'(k)$ must increase for every c, therefore k must fall and $\dot{c} = 0$ locus shifts to the left (see Figure 3).