Economics 202A
Suggested Solutions to Problem Set 5

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1 Romer, 3.1.

Our R&D model without a capital is

$$\begin{aligned} Y(t) &= A(t)(1-a_L)L(t) && (1)\\ \dot{A}(t) &= B[a_L L(t)]^\gamma A(t)^\theta,\ \theta<1. && (2)\end{aligned}$$

(a) Define g_A^* a BGP growth rate of A: $\dot{A}(t) = g_A^* A(t)$. Recall (or derive) that in the model without capital when $\theta < 1$, $g_A^* = \frac{\gamma}{1-\theta}n$. Therefore $\dot{A}(t) = \frac{\gamma}{1-\theta}nA(t)$. But $\dot{A}(t)$ is also defined by the equation (2), therefore

$$\begin{aligned} \frac{\gamma}{1-\theta}nA(t) &= B[a_L L(t)]^\gamma A(t)^\theta \\ A(t)^{1-\theta} &= \frac{Ba_L^\gamma L(t)^\gamma(1-\theta)}{n\gamma} \\ A(t) &= \left[\frac{Ba_L^\gamma L(t)^\gamma(1-\theta)}{n\gamma}\right]^{\frac{1}{1-\theta}}. \end{aligned}$$

(b) Now substitute the result of (a) into (1):

$$Y(t) = \left[\frac{Ba_L^\gamma L(t)^\gamma(1-\theta)}{n\gamma}\right]^{\frac{1}{1-\theta}}(1-a_L)L(t).$$

We now have to maximize this Y with respect to a_L. Note that $\text{argmax} f(x) = \text{argmax} \ln f(x)$, therefore we can just maximize $\ln Y$ with respect to a_L.

$$\begin{aligned}\ln Y &= \frac{1}{1-\theta}[\ln B + \gamma \ln a_L + \gamma \ln L + \ln(1-\theta) - \ln n - \ln \gamma] + \\ &+ \ln(1-a_L) + \ln L\end{aligned}$$

The F.O.C. is

$$\frac{\gamma}{(1-\theta)a_L} = \frac{1}{1-a_L},$$

$$a_L = \frac{\gamma}{1-\theta+\gamma}.$$

The higher is the return on knowledge in R&D sector (θ) and the higher is the return on labor in R&D sector, the more people should be employed in R&D sector in order to maximize the balanced growth path level of output.

2 Romer, 3.10.

This is a version of a famous North-South set up you might have heard of. We have two economies described by identical R&D models except for the way knowledge is accumulated.

$$\begin{aligned}Y_i &= K_i^{\alpha}[A_i(1-a_{Li})L_i]^{1-\alpha}, &(3)\\ \dot{K}_i &= s_i Y_i,\ i = N, S. &(4)\end{aligned}$$

In the North the knowledge is produced

$$\dot{A}_N = B a_{LN} L_N A_N, \tag{5}$$

in the South the knowledge is copied

$$\dot{A}_S = \begin{cases} \mu a_{LS} L_S [A_N - A_S], & A_N > A_S \\ 0 & \text{otherwise} \end{cases}$$

and there is no population growth in either country.

(a) The North is described by our standard R&D model with no population growth and $\theta = 1$. Therefore we know that the growth rate of output per worker is equal to $Ba_{LN}L_N$.[1]

$Z(t) = \frac{A_S(t)}{A_N(t)}$ is the share of goods already copied in the South. The growth rate of Z is

$$\frac{\dot{Z}}{Z} = \frac{\dot{A}_S}{A_S} - \frac{\dot{A}_N}{A_N} \equiv g_{AS} - gAN.$$

We already know that $g_{AN} = Ba_{LN}L_N$, from the definition of $\dot{A}_S$

$$\begin{aligned} g_{AS} &= \frac{\mu a_{LS} L_S A_N}{A_S} - \mu a_{LS} L_S = \\ &= \frac{\mu a_{LS} L_S}{Z} - \mu a_{LS} L_S, \\ \dot{Z} &= \mu a_{LS} L_S - \mu a_{LS} L_S Z - Ba_{LN} L_N Z = \\ &= \mu a_{LS} L_S - (\mu a_{LS} L_S + Ba_{LN} L_N) Z. \end{aligned}$$

We can now plot the dynamics of Z on the phase diagram (see Figure 1). Note that all our derivations above are good for $Z < 1$, when not all the knowledge is copied by the South. If all the knowledge is copied then $Z = 1$, but this can not be a BGP, since in the next period A_N will increase, but A_S will not, therefore Z will fall and the above derivation applies. The assumptions about the knowledge accumulation process in the South suggests that there is no way Z can be greater then 1. Thus restricting attention to the case when $Z < 1$ is OK in this case.

We can see that steady state of Z is stable since $\dot{Z}$ is a decreasing function

[1] Since the population is constant, the growth rate of output per worker is equal to the growth rate of output. From (4) we know that on the BGP growth rate of capital should be equal to the growth rate of output. If we substitute (3) into (4) we get

$$g_{KN} = s_N \left[\frac{A_N(1 - a_{LN})L_N}{K_N} \right]^{1-\alpha}$$

and since L_N is constant, the growth rate of capital should be equal to the growth rate of knowledge on the BGP, but growth rate of knowledge is easily determined from (5): $g_{AN} = Ba_{LN}L_N$.

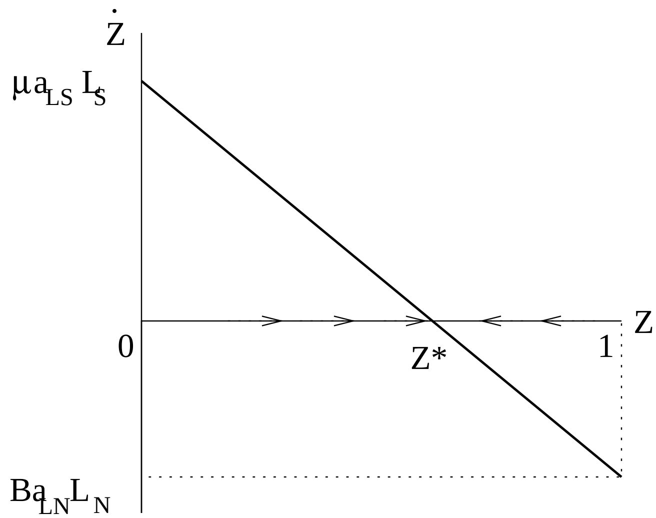

Figure 1: The dynamics of Z

of Z. What does Z converge to? $\dot{Z} = 0$ when

$$Z^* = \frac{\mu a_{LS} L_S}{\mu a_{LS} L_S + B a_{LN} L_N} < 1,$$

which is the BGP level of Z.

Since Z is constant on the BGP, $g_{AS} = g_{AN} = B a_{LN} L_N$. Since the rest of the South economy is described by exactly the same model, using the same argument as in (a) we can conclude that the growth rate of output per worker in the South is equal to the growth rate of output per worker in the North.

$$g_{\frac{Y}{L}S} = g_{YS} = g_{KS} = g_{AS} = g_{AN} = B a_{LN} L_N.$$

(c) We just concluded that the growth rates in two countries are the same on the BGP. What about the level of their wealth? Suppose that except for knowledge accumulation process, the countries are identical: $a_{LN} = a_{LS} = a_L$, $s_N = s_S = s$. In this case we can write

$$\frac{Y_S}{Y_N} = \frac{K_S^{\alpha}[A_S(1-a_L)L_S]^{1-\alpha}}{K_N^{\alpha}[A_N(1-a_L)L_N]^{1-\alpha}} =$$

$$
\begin{aligned}
\frac{Y_S}{Y_N} &= \frac{s^\alpha Y_S^\alpha A_S^{1-\alpha}[(1-a_L)L_S]^{1-\alpha}}{s^\alpha Y_N^\alpha A_N^{1-\alpha}[(1-a_L)L_N]^{1-\alpha}}, \\
\left[\frac{Y_S}{Y_N}\right]^{1-\alpha} &= Z^{1-\alpha}\left[\frac{L_S}{L_N}\right]^{1-\alpha}, \\
\frac{Y_S}{Y_N} &= Z\frac{L_S}{L_N}, \\
\frac{Y_S/L_S}{Y_N/L_N} &= Z.
\end{aligned}
$$

But on the BGP

$$
Z = \frac{\mu a_{LS} L_S}{\mu a_{LS} L_S + B a_{LN} L_N},
$$

therefore

$$
\frac{Y_S/L_S}{Y_N/L_N} = \frac{\mu a_{LS} L_S}{\mu a_{LS} L_S + B a_{LN} L_N}.
$$

There are two things to note about this result. First is that the level of income per worker in the South is lower than in the North. Second is that relative output per worker in the south positively depends on the number of workers in the R&D sector in the South. The more people work in the "copying" sector of the South, the smaller is the gap in the wealth of North and South.

3 Romer, 3.11.

(a) This is a different North-South R&D model. Here the process of technology diffusion is described by the lag with which the technology of the North can be used in the South.

$$
\begin{aligned}
Y_N(t) &= A_N(t)(1-a_L)L_N, \\
\dot{A}_N(t) &= a_L L_N A_N(t); \\
Y_S(t) &= A_S(t)L_S, \\
A_S(t) &= A_N(t-\tau).
\end{aligned}
$$

We will do the quantitative exercise to see whether this kind of lag can explain well the income differences. So assume that $g_{\frac{Y}{L}N} = g_{YN} = 3\%$ per year. How large should be the lag in the knowledge transmission for the output per worker in the North to be 10 times as high as the output per worker in the South?

We need to find τ such that $\frac{Y_N/L_N}{Y_S/L_S} = 10$. From the production function of the North we know

$$\frac{Y_N}{L_N} = A_N(1 - a_L),$$

and therefore the growth rate of output per worker in the North is equal to the rate of growth of the knowledge and we know that it is 0.03 per year. Therefore we can write that[2]

$$A_N(t) = \mathrm{e}^{0.03\tau} A_N(t - \tau),$$

thus

$$A_N(t) = \mathrm{e}^{0.03\tau} A_S(t).$$

We also know from the South production function that

$$\frac{Y_S}{L_S} = A_S.$$

Combining these results and taking into account the assumption that a_L is close to zero[3], we get

$$\frac{Y_N/L_N}{Y_S/L_S} = \frac{A_N(t)(1 - a_L)}{A_S(t)} \approx \frac{A_N(t)}{A_S(t)} = \mathrm{e}^{0.03\tau},$$

which we want to be equal to 10. Using calculator we can find that $0.03\tau = \ln(10)$ when $\tau = 76.75$ years. This is way too high to be realistic in describing the transmission of technology. Therefore this model does not do a good job in describing cross-country income differences.

(b) Let's see whether Solow model will do better job with the same assumption on the technology transmission. So now our two economies are described by the model

$$y_N \ = \ f(k_N),$$

[2]Using the solution to the world's simplest differential equation.

[3]Note that the increase in a_L will lead to even higher τ required to get the 10 years gap, therefore it seems to be an OK assumption to make. To convince yourself, try $a_L = 0.5$.

$$\begin{aligned}
\dot{K}_N &= sY_N - \delta K_N, \\
\dot{L}_N &= nL_N, \\
\dot{A}_N &= gA_N; \\
y_S &= f(k_S), \\
\dot{K}_S &= sY_S - \delta K_S, \\
\dot{L}_S &= nL_S, \\
\dot{A}_S(t) &= A_N(t-\tau).
\end{aligned}$$

(i) First we have to show that k^* is the same for North and South. For the North k_N^* is defined by

$$sf(k_N^*) = (n+g+\delta)k_N^*,$$

because the economy of the North is described by the Solow model. For the south the only difference is the knowledge accumulation. As we derived in part (a),

$$A_N(t) = \mathrm{e}^{g\tau} A_S(t)$$

and since τ is constant, $A_S(t)$ must grow at the same rate g as $A_N(t)$. Therefore for the South k_S^* is defined by the same equation

$$sf(k_S^*) = (n+g+\delta)k_S^*.$$

We know that k^* is uniquely defined from this equation (result of Inada conditions) and therefore k_S^* must equal to k_N^*.

(ii) The result in (i) hints us the answer to (ii): since k on the BGP is the same for both countries, the introduction of capital should not make any difference. Since production functions are the same, output per unit of effective labor will be also equal in both countries on the BGP: $y_N^* = y_S^*$. Thus

$$\frac{Y_N/L_N}{Y_S/L_S} = \frac{A_N(t)y_N^*}{A_S(t)y_S^*} = \frac{A_N(t)}{A_S(t)} = \mathrm{e}^{0.03\tau}$$

as we assumed the same growth rate as in (a). Therefore the lag still has to be 76.75 years in order to explain the 10-fold difference in the per-capita income.