Advanced Microeconomics
(Economics 104)
Fall 2011
Extensive games with imperfect information

Topics

- Formalities.
- Sequential equilibrium.

Preliminaries (O 10.1, OR 11.1)

Players are always (somewhat) imperfectly informed when making their decisions. Now, we allow players also to have partial information about the history of actions. Therefore, we need to

- generalize the definition of extensive game with perfect information, and
- extend the notion of SPE to extensive games with imperfect information.

Recall that an extensive game with perfect information

$$\Gamma = \langle N, H, P, (\succsim_i) \rangle$$

consists of

- A set N of players.
- A finite or infinite set H of sequences (histories), each component an action taken by a player.
- A player function $P : H \backslash Z \to N$ such that $P(h)$ being the player who takes an action after history h.
- A preference relation $\succsim_i$on Z for each player $i \in N$.

In addition, an extensive game with imperfect information

$$\Gamma = \langle N, H, P, f_c, (\mathcal{I}_i)_{i\in N}, (\succsim_i)\rangle$$

consists of

- $P(h) = c$ indicating that chance determines the action taken after the history h, and for any h such that $P(h) = c$ a probability measure $f_c(\cdot\,|h\,)$ on $A(h)$.
- A partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$ for every $i \in N$ such that

  $$A(h) = A(h')$$

  whenever h and h' are in the same partition.

$\mathcal{I}_i$ is an information partition, and $I_i \in \mathcal{I}_i$ is an information set (any $h \in I_i$ is indistinguishable).

Thus, an extensive game with imperfect information

$$\langle N, H, P, f_c, (\mathcal{I}_i)_{i\in N}, (\succsim_i)\rangle$$

in which for every player all information sets are singletons

$$\langle N, H, P, f_c, (\succsim_i)\rangle$$

is an extensive game with perfect information.

Each player's information partition is the primitive of the game. A (pure) strategy is a function that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_i$.

Pure, mixed and behavioral strategies (O 10.2-3, OR 11.4)

In an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i)_{i \in N}, (\succsim_i)\rangle$, for player $i \in N$

- A pure strategy is a function that assigns and action $a_i \in A(I_i)$ to each information set $I_i \in \mathcal{I}_i$.
- A mixed strategy is a probability measure over the set of pure strategies.
- A behavioral strategy is a collection

  $$(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$$

  of independent probability measures, where $\beta_i(I_i)$ is a probability measure over $A(I_i)$.

For any $\sigma = (\sigma_i)_{i \in N}$ (mixed or behavioral) an outcome $O(\sigma)$ is a probability distribution over $z \in Z$ that results from σ.

Two strategies (mixed or behavioral) of player i, σ_i and σ_i', are *outcome equivalent* if

$$O(\sigma_i, s_{-i}) = O(\sigma_i', s_{-i})$$

for every collection s_{-i} of pure strategies.

In any finite game with perfect recall, any mixed strategy of a player has an outcome-equivalent behavioral strategy)the converse is true for a set of games that includes all those with perfect recall).

Sequential equilibrium

Some intuition

- consider the games depicted by OR in figures 219.1 and 220.1. In these games, we can wonder about choices made off the (Nash) equilibrium path. For example, (LR) is a NE in 219.1.
- But, in both player 2's choice is not "reasonable." (why?) In these cases the off-equilibrium path of play is "suspicious."

The spirit of SPE

- players should choose optimally also in off-the-equilibrium information sets. But, subgame perfection is no help because there are no proper subgames, and need to evaluate ex post expected payoffs.

The problem

- Beliefs are not always computable from the strategies via Bayes' rule. As in (219.1), sometimes this does not matter (why?), but it is not true in general.

Strategies and beliefs (O 10.4, OR 12.1)

The notion of sequential equilibrium is meant to address these problems.

A sequential equilibrium consists of two pieces:

- a profile of behavioral strategies β, and
- a system of beliefs μ.

β prescribes for every information set a probability distribution over the actions available.

That is, for each i, $h \in I_i \in \mathcal{I}_i$ and $a \in A(I_i)$, $\beta_i(I_i)(a)$ is the probability assigned by β_i to a.

μ assigns to every information set a probability distribution over the histories that lead to the set.

That is, μ is a function from the set of histories H to $[0, 1]$ such that for any i and I_i, $\sum_{h \in I_i} \mu(h) = 1$.

Roughly speaking,

A sequential equilibrium is a strategy profile β and beliefs μ which is

- sequentially rational, and
- consistent.

Sequentially rational - starting from every information set, players play optimally given that

- what has transpired previously is given by μ, and
- what will transpire subsequently is given by β.

Consistency - beliefs and strategies make sense together at least at level of Bayes' rule in the sense that

- beliefs are computed from strategies in a Bayesian way.

But, Bayes' rule does not apply to information sets that are not reached with positive probability on the path of play...

Formalities (OR 12.2-3)

(β, μ) is sequentially rational if for each $i \in N$ and every $I_i \in \mathcal{I}_i$

$$O(\beta, \mu \,|I_i\,) \succsim_i O((\beta_{-i}, \beta_i'), \mu \,|I_i\,)$$

for every strategy β_i'.

(β, μ) is consistent if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ such that for each n

- β^n is completely (strictly) mixed,
- μ^n is derived from β^n using Bayes' rule, and
- the limit of the vector $((\beta^n, \mu^n))_{n=1}^{\infty}$ in n is (β, μ).

(β, μ) is a sequential equilibrium if it is sequentially rational and consistent.