University of California – Berkeley
Department of Economics
Game Theory in the Social Sciences
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Lecture III
Nash equilibriaum

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Strategic games
(review)

A two-player (finite) strategic game

The game can be described conveniently in a so-called bi-matrix. For example, a generic 2×2 (two players and two possible actions for each player) game

$$\begin{array}{c|c|c|} & L & R \\ \hline T & a_1, a_2 & b_1, b_2 \\ \hline B & c_1, c_2 & d_1, d_2 \\ \hline \end{array}$$

where the two rows (resp. columns) correspond to the possible actions of player 1 (resp. 2). The two numbers in a box formed by a specific row and column are the players' payoffs given that these actions were chosen.

In this game above a_1 and a_2 are the payoffs of player 1 and player 2 respectively when player 1 is choosing strategy T and player 2 strategy L.

Applying the definition of a strategic game to the 2×2 game above yields:

- Players: $\{1, 2\}$
- Action sets: $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$
- Action profiles (outcomes):

$$A = A_1 \times A_2 = \{(T, L), (T, R), (B, L), (B, R)\}$$

- Preferences: $\succsim_1$ and $\succsim_2$ are given by the bi-matrix.

Dominance I

	b_1	b_2	b_3
a_1	1, _	1, _	1, _
a_2	1, _	0, _	1, _
a_3	0, _	0, _	0, _

For player 1, action a_2 is weakly dominated by a_1, and action a_3 is weakly dominated by a_2 and strictly dominated by a_1.

Dominance II

	b_1	b_2	b_3
a_1	$4,3$	$5,1$	$6,2$
a_2	$2,1$	$8,4$	$3,6$
a_3	$3,0$	$9,6$	$2,8$

	b_1	b_3
a_1	$4,3$	$6,2$
a_2	$2,1$	$3,6$
a_3	$3,0$	$2,8$

	b_1	b_3
a_1	$4,3$	$6,2$

$\Longrightarrow$ by iterated elimination of strictly dominated strategies.

Rationalizability

	b_1	b_2	b_3	b_4
a_1	$0, 7$	$2, 5$	$7, 0$	$0, 1$
a_2	$5, 2$	$3, 3$	$5, 2$	$0, 1$
a_3	$7, 0$	$2, 5$	$0, 7$	$0, 1$
a_4	$0, 0$	$0, -2$	$0, 0$	$10, -1$

The rationalizable actions are a_1, a_2, a_3 for player 1 and b_1, b_2, b_3 for player 2.

Classical 2×2 games

- The following simple 2×2 games represent a variety of strategic situations.

- Despite their simplicity, each game captures the essence of a type of strategic interaction that is present in more complex situations.

- These classical games "span" the set of almost *all* games (strategic equivalence).

Game I: Prisoner's Dilemma

	$Work$	$Goof$
$Work$	$3, 3$	$0, 4$
$Goof$	$4, 0$	$1, 1$

A situation where there are gains from cooperation but each player has an incentive to "free ride."

Examples: team work, duopoly, arm/advertisement/R&D race, public goods, and more.

Game II: Battle of the Sexes (BoS)

	$Ball$	$Show$
$Ball$	$2, 1$	$0, 0$
$Show$	$0, 0$	$1, 2$

Like the Prisoner's Dilemma, Battle of the Sexes models a wide variety of situations.

Examples: political stands, mergers, among others.

Game III-V: Coordination, Hawk-Dove, and Matching Pennies

	$Ball$	$Show$
$Ball$	2, 2	0, 0
$Show$	0, 0	1, 1

	$Dove$	$Hawk$
$Dove$	3, 3	1, 4
$Hawk$	4, 1	0, 0

	$Head$	$Tail$
$Head$	1, −1	−1, 1
$Tail$	−1, 1	1, −1

Best response and dominated actions

Action T is player 1's *best response* to action L player 2 if T is the optimal choice when 1 *conjectures* that 2 will play L.

Player 1's action T is *strictly* dominated if it is never a best response (inferior to B no matter what the other players do).

In the Prisoner's Dilemma, for example, action $Work$ is strictly dominated by action $Goof$. As we will see, a strictly dominated action is not used in any Nash equilibrium.

Nash equilibrium

Nash equilibrium (NE) is a steady state of the play of a strategic game – no player has a profitable deviation given the actions of the other players.

Put differently, a NE is a set of actions such that all players are doing their best given the actions of the other players.

Mixed strategy Nash equilibrium in the BoS

Suppose that, each player can randomize among all her strategies so choices are not deterministic:

$$
\begin{array}{cc|c|c|}
 & & q & 1-q \\
 & & L & R \\
\hline
p & T & pq & p(1-q) \\
\hline
1-p & B & (1-p)q & (1-p)(1-q) \\
\hline
\end{array}
$$

Let p and q be the probabilities that player 1 and 2 respectively assign to the strategy *Ball*.

Player 2 will be indifferent between using her strategy B and S when player 1 assigns a probability p such that her expected payoffs from playing B and S are the same. That is,

$$\begin{gathered} 1p + 0(1-p) = 0p + 2(1-p) \\ p = 2 - 2p \\ p^* = 2/3 \end{gathered}$$

Hence, when player 1 assigns probability $p^* = 2/3$ to her strategy B and probability $1 - p^* = 1/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

Similarly, player 1 will be indifferent between using her strategy B and S when player 2 assigns a probability q such that her expected payoffs from playing B and S are the same. That is,

$$2q + 0(1-q) = 0q + 1(1-q)$$
$$2q = 1 - q$$
$$q^* = 1/3$$

Hence, when player 2 assigns probability $q^* = 1/3$ to her strategy B and probability $1 - q^* = 2/3$ to her strategy S, player 2 is indifferent between playing B or S any mixture of them.

In terms of best responses:

$$B_1(q) = \begin{cases} p = 1 & if \quad p > 1/3 \\ p \in [0,1] & if \quad p = 1/3 \\ p = 0 & if \quad p < 1/3 \end{cases}$$

$$B_2(p) = \begin{cases} q = 1 & if \quad p > 2/3 \\ q \in [0,1] & if \quad p = 2/3 \\ q = 0 & if \quad p < 2/3 \end{cases}$$

The BoS has two Nash equilibria in pure strategies $\{(B, B), (S, S)\}$ and one in mixed strategies $\{(2/3, 1/3)\}$. In fact, any game with a finite number of players and a finite number of strategies for each player has Nash equilibrium (Nash, 1950).

Three Matching Pennies games in the laboratory

		.48	.52
		a_2	b_2
.48	a_1	80, 40	40, 80
.52	b_1	40, 80	80, 40

		.16	.84
		a_2	b_2
.96	a_1	320, 40	40, 80
.04	b_1	40, 80	80, 40

		.80	.20
		a_2	b_2
.08	a_1	44, 40	40, 80
.92	b_1	40, 80	80, 40

Food for thought

LUPI

Many players simultaneously chose an integer between 1 and 99,999. Who-ever chooses the lowest unique positive integer (LUPI) wins.

Question What does an equilibrium model of behavior predict in this game?

The field version of LUPI, called Limbo, was introduced by the government-owned Swedish gambling monopoly Svenska Spel. Despite its complexity, there is a surprising degree of convergence toward equilibrium.

Morra

A two-player game in which each player simultaneously hold either one or two fingers and each guesses the total number of fingers held up.

If exactly one player guesses correctly, then the other player pays her the amount of her guess.

Question Model the situation as a strategic game and describe the equilibrium model of behavior predict in this game.

The game was played in ancient Rome, where it was known as “micatio.”

Maximal game
(sealed-bid second-price auction)

Two bidders, each of whom privately observes a signal X_i that is independent and identically distributed (i.i.d.) from a uniform distribution on $[0, 10]$.

Let $X^{\text{max}} = \max\{X_1, X_2\}$ and assume the ex-post common value to the bidders is X^{max}.

Bidders bid in a sealed-bid second-price auction where the highest bidder wins, earns the common value X^{max} and pays the second highest bid.