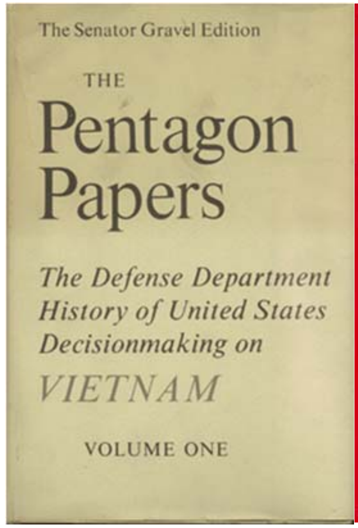
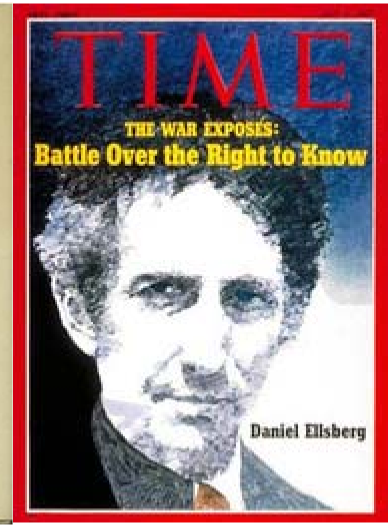
University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
Fall 2023

Preferences, Utility, and Choice
(Kreps Ch. 1 and Rubinstein Ch. 1-3)

Aug 31 and Sep 5

Ever since Ellsberg...


The Senator Gravel Edition
THE
Pentagon
Papers
The Defense Department
History of United States
Decisionmaking on
VIETNAM
VOLUME ONE


TIME
THE WAR EXPOSES:
Battle Over the Right to Know
Daniel Ellsberg


The Washington Post
FINAL
Documents Reveal U.S. Effort
In '54 to Delay Viet Election
Times Lists
But Retains
Viet Papers
Times' Stand Wins
Wide Hill Support
FBI Seeking
Ex-Pentagon
Aide in Leak
Prosecutor Links Bobby Baker
To Ring Accused in NW Thefts

Experiments à la Ellsberg

Consider the following four two-color Ellsberg-type urns (Halevy, 2007):

I. 5 red balls and 5 black balls

II. an unknown number of red and black balls

III. a bag containing 11 tickets with the numbers 0-10; the number written on the drawn ticket determines the number of red balls

IV. a bag containing 2 tickets with the numbers 0 and 10; the number written on the drawn ticket determines the number of red balls

Kreps' word of advice...

A recommendation for this course which extends to any microeconomics book/paper (mathematical in character):

$\Longrightarrow$ Read carefully and slowly for details – follow the details of the proofs one step at a time, have a pad and pen, make notes, and finish arguments.

$\Longrightarrow$ But do not lose the "plot line," which is just as important – What is the framework? What are the results? How do the results tie together?

Constructing (mathematical) proofs is a skill you learn best – and perhaps only – by doing.

Outline

Preferences

The preferences of the $\mathcal{DM}$ for $x, y \in X$ (a finite set of objects of choice) are specified by a binary relation $\succsim$ where $x \succsim y$ is read "x is at least as good as y" or as "x is weakly preferred to y."

- $\succsim$ on X is complete if for $\forall x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both).
- $\succsim$ on X is is transitive if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

$\Longrightarrow$ It would be nice to begin with something less abstract / more concrete but $\succsim$ are the logical starting point for choice theory...

Utility

The $\mathcal{DM}$'s utility function is a real-valued function $u : X \to \mathbb{R}$. It represents $\succsim$ if

$$x \succsim y \Longleftrightarrow u(x) \geq u(y), \tag{1}$$

that is, the $\mathcal{DM}$ regards items of higher utility as better.

$\Longrightarrow$ Many different utility functions represents the same preferences so utility numbers (for now) have only ordinal and not cardinal meaning.

Choice

A choice function c such that $c(A) \subseteq A$ which specifies for each nonempty subset $A \subseteq X$ what the $\mathcal{DM}$ would be 'content' to have.

– c is generated by $\succsim$ if $\forall A$

$$c(A) = \{x \in A : x \succsim y \ \forall y \in A\}. \tag{2}$$

– c is generated by utility maximization of u if $\forall A$

$$c(A) = \{x \in A : u(x) \geq u(y) \ \forall y \in A\}. \tag{3}$$

Properties of choice functions

C1 A choice function c satisfies finite nonemptiness if $c(A) \neq \emptyset$ for $\forall A \in \mathcal{A}$ where $\mathcal{A}$ is all the nonempty subsets $A \subseteq X$.

C2 A choice function c satisfies choice coherence if $\forall x, y \in X$ and $\forall A, B \in \mathcal{A}$,

$$\text{if } x, y \in A \cap B,\ x \in c(A), \text{ and } y \notin c(A), \text{ then } y \notin c(B).$$

The main result(s) (X finite)

- If c satisfies C1 and C2, then there exist both a complete and transitive $\succsim$ and u that 'produce' choices according to c via (2) and (3), respectively.

- If $\succsim$ is complete and transitive, then the c it produces via (2) satisfies C1 and C2, and there exists u that represents $\succsim$ via (1).

- Given any u, the c it produces via (3) satisfies C1 and C2 and the $\succsim$ it represents via (1) is complete and transitive. And the c produced by that $\succsim$ via (2) is precisely the c produced from u via (3).

- In words, choice behavior that satisfies C1 and C2 is equivalent to complete and transitive $\succsim$-optimality and both of which are equivalent to u-maximization.

- This conglomerate (the two pairs of assumptions) is the standard model of choice in microeconomics:

  $\longrightarrow$ prove and generalize
  $\longrightarrow$ extend to infinite X
  $\longrightarrow$ comments, extensions, variations, and criticisms...

$\Longrightarrow$ Well, we know where we're goin' But we don't know where we've been... (Talking Heads |Little Creatures)

Preferences

We start by taking up the following (simple) story:

– There is a (finite) set X of options and a $\mathcal{DM}$ who is willing to express her/his preferences among these options by making paired comparisons.

– We want to 'fully' describe the preferences of the $\mathcal{DM}$ toward the options in X, thinking about preferences independently of choice.

$\Longrightarrow$ Your attitude toward the PhD programs you have applied to before finding out to which of them you have been accepted...

Rubinstein's (imaginary) questionnaires

Questionnaire $Q(x, y)$: Check one of the following three boxes:

- $\square$ I prefer x to y ($x \succ y$)
- $\square$ I prefer y to x ($y \succ x$)
- $\square$ I am indifferent (I)

– I cannot compare, do not know, have no opinion, prefer both, need to consult a friend (and so on) are "illegal" answers...

– We use the symbol I for indifference because

$$x \sim y \Longleftrightarrow x \succsim y \text{ and } y \succsim x.$$

For "legal" answers to the questionnaire Q to qualify as preferences must satisfy two "consistency" requirements:

1. No "framing effect" – the answer to $Q(x, y)$ must be the answer to $Q(y, x)$.

2. The answers to $Q(x, y)$ and $Q(y, z)$ are consistent (in the transitive sense...) with the answer to $Q(x, z)$.

And the answer to the "silly" question $Q(x, x)$ must be I am indifferent (I) but we will consider only distinct options $x \neq y$.

Preferences (based on questionnaire Q): Preferences are a function f that assigns to any $x, y \in X$ one of the three "values" $x \succ y$, $y \succ x$, or I so that for any $x, y, z \in X$, the following two properties hold:

1. No order effect: $f(x, y) = f(y, x)$.

2. Transitivity:

$$\begin{aligned} f(x,y) &= x \succ y \text{ and } f(y,z) = y \succ z \Longrightarrow f(x,z) = x \succ z \\ f(x,y) &= I \text{ and } f(y,z) = I \Longrightarrow f(x,z) = I. \end{aligned}$$

All other consistency requirements such as $f(x, y) = x \succ y$ and $f(y, z) = I \Longrightarrow f(x, z) = x \succ z$ follow from the above conditions.

Questionnaire $R(x, y)$: Is $x \succsim y$? Check one (and only one) of the following two boxes:

☐ Yes
☐ No

If the answer to $R(x, y)$ is "Yes" then we identify the response to this with $x \succsim y$. To qualify as preferences, responses must satisfy two conditions:

1. At least one answer to $R(x, y)$ and $R(y, x)$ as well as the answer to $R(x, x)$ must be "Yes."

2. For any $x, y, z \in X$, if the answers to $R(x, y)$ and $R(y, z)$ are "Yes" then so is the answer to $R(x, z)$.

Preferences (based on questionnaire R): Preferences are a binary relation $\succsim$ satisfying:

1. Reflexivity: $\forall x \in X$, $x \succsim x$.

2. Completeness: $\forall x, y \in X$, $x \succsim y$ and/or $y \succsim x$.

3. Transitivity: $\forall x, y, z \in X$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

This is the conventional definition of preferences but the two definitions of preferences are equivalent.

The equivalence of the two definitions of preferences

$Q(x,y)$ and $Q(y,x)$	$R(x,y)$ and $R(y,x)$
$x \succ y$	Yes No
$y \succ x$	No Yes
I	Yes Yes

- Equivalence of two definitions (in microeconomics) requires the existence of a one-to-one (bijection) correspondence that preserves the interpretation...

- Exercise: Construct a "translation" between answers to Q (qualify as preferences by the 1st definition) and answers to R (qualify as preferences by the 2nd definition).

Sources of intransitivity

(1) Aggregating of considerations:

- Let $X = \{a, b, c\}$ and assume the $\mathcal{DM}$'s attitude is $x \succ y$ if the majority of considerations supports x.

- The aggregation of three considerations such that

$$a \succ_1 b \succ_1 c \quad b \succ_2 c \succ_2 a \quad \text{and} \quad c \succ_3 a \succ_3 b$$

  leads to violating transitivity.

(2) Using similarities:

- Let let $X = \mathbb{R}$ and assume the $\mathcal{DM}$'s attitude is "the larger the better" but $\mathcal{DM}$ is unable to determine whether $a > b$ unless $|a - b| \geq 1$, thus

$$x \succ y \text{ if } x \geq y + 1$$

- This is not a preference relation because

$$0 \sim \frac{2}{3} \quad \text{and} \quad \frac{2}{3} \sim \frac{4}{3} \quad \text{but not} \quad 0 \sim \frac{4}{3}.$$

Utility

It is possible to avoid the notion of a utility representation and to "do economics" solely based on the notion of preferences, but...

$\Longrightarrow$ $u : X \to \mathbb{R}$ represents the preference $x \succsim y$ if $\forall x, y \in X$

$$x \succsim y \Longleftrightarrow u(x) \geq u(y).$$

If the function u represents the preference relation $\succsim$, we refer to it as a utility function and say that $\succsim$ has a utility representation.

If u represents $\succsim$, then for any strictly increasing function $f : \mathbb{R} \longrightarrow \mathbb{R}$, the function $v(x) = f(u(x))$ also represents $\succsim$.

$$
\begin{array}{rcl}
a & \succsim & b \\
 & \Updownarrow & \\
u(a) & \geq & u(b) \\
 & \Updownarrow & \\
f(u(a)) & \geq & f(u(b)) \\
 & \Updownarrow & \\
v(a) & \geq & v(b)
\end{array}
$$

If $\succsim$ has a utility representation, then it has an infinite number of such representations...

Existence of a utility representation

X is finite:

– If $\succsim$ is a preference relation on a finite set X, then $\succsim$ has a utility representation with values that are natural numbers.

X is countable (if infinite has a one-to-one correspondence with the set of natural numbers):

– If $\succsim$ is a preference relation on a countable set X, then $\succsim$ has a utility representation with a range $(0, 1)$.

Proof (for finite X):

First we show by induction that any finite set $A \subseteq X$, has a minimal/maximal element – $a \in X$ is minimal if $a \precsim x$ for any $x \in X$.

– If A is a singleton, then by reflexivity its (single) element is minimal.

– By the inductive assumption, the set $A - \{x\}$ for some $x \in A$ has a minimal element, say y.

– If $x \succsim y$, then y is minimal in A. If $y \succsim x$, then by transitivity $z \succsim x$ for all $z \in A - \{x\}$ so x is minimal.

Next let X_{k+1} be set of minimal elements in $X - X_1 - \cdots - X_k$ and note that $X_{k+1} \neq \emptyset$.

- Let $u(x) = k$ if $x \in X_k$ (x is "eliminated" after k steps) and note that because X is finite $u(x) \leq |X|$.

- If $a \succ b$ then $a \notin X_1 \cup X_2 \cup \cdots \cup X_{u(b)}$ so $u(a) > u(b)$ and if $a \sim b$ then $u(a) = u(b)$.

$\Longrightarrow$ Hence, u represents $\succsim$. The proof of the case where X is countable is left as an exercise. No further assumptions on $\succsim$ are needed.

Lexicographic preferences

"...even though they are derived from a simple and commonly used procedure," lexicographic preferences do not have a utility representation by $u : X \to \mathbb{R}$.

Let $X = [0,1] \times [0,1]$ (the unit square) and let $x \succsim_k y$ if $x_k \geq y_k$. The lexicographic preferences $\succsim_L$ induced by $\succsim_1$ and $\succsim_2$ are given by

$$(a_1, a_2) \succsim_L (b_1, b_2) \text{ if } a_1 > b_1 \text{ or } a_1 = b_1 \text{ and } a_2 \geq b_2.$$

Proof: Assume $u : X \to \mathbb{R}$ represents $\succsim_L$toward contradiction.

– For any $a \in [0,1]$, $(a,1) \succ (a,0)$ and thus $u(a,1) > u(a,0)$. Let $q(a)$ be a rational number in the nonempty interval

$$I_a = (u(a,0), u(a,1)).$$

– $q(a) : [0,1] \to \mathbb{R}$ and it is one-to-one function since

$$b > a \Longrightarrow (b,0) \succ_L (a,1) \Longrightarrow u(b,0) > u(a,1)$$

– It follows that the intervals I_a and I_b are are disjoint so $q(a) \neq q(b)$ but the cardinality of the rational numbers is lower than that of the continuum.

Continuity of preferences

Continuity condition guarantees the existence of a utility representation when X is an infinite subset of a Euclidean space (Debreu's theorem).

I. $\succsim$ on X is continuous if whenever $a \succ b$, there are (small) balls B_a and B_b around a and b, respectively, such that $a \succ b$ for all $x \in B_a$ and $y \in B_b$.

II. $\succsim$ on X is continuous if $\{(a_n, b_n)\}$ a sequence of pairs satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$, then $a \succsim b$. That is, if the graph of $\succsim$

$$\{(x, y) \mid x \succsim y \subseteq X \times X\}$$

is closed.

Three remarks:

1. $\succsim$ on X satisfies (I) iff it satisfies (II).

2. If $\succsim$ can be represented by a continuous u then it is continuous.

3. $\succsim_L$is not continuous: for "small" ε

$$(1,1) \succ_L (1,0) \text{ but } (1-\varepsilon,1) \prec_L (1,0).$$

$\Longrightarrow$ Prove (1) and (2) as an exercise.


Brian Newton
James Peoples
Cannot remember
Chuck Mason
Donald Andrews
Frank Rimlinger
Bob Jacobson
Coach Debreu
Mike Luger
Bob Powell
John Henley
Curt Grimm
Stuart Gabriel
Phil Young
Spence Krane
Ken "MadDog" Meyers
Barry Ickes
Larry Jones
ECONOMICS
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Debreu's (1954, 1960) theorem

One of the classical results in economic theory:

> **Debreu's theorem** Let $\succsim$ be a continuous preference relation on X, which is a convex subset of $\mathbb{R}^n$. Then $\succsim$ has a continuous utility representation.

The proof (in Rubinstein), relies on the mathematical concept of a dense set – a set Y is said to be dense in X if every non-empty open set $B \subset X$ contains an element in Y. Any set $X \subseteq \mathbb{R}^n$ has a countable dense subset.

Choice

A choice function c such that $c(A) \subseteq A$ which specifies for each nonempty subset $A \subseteq X$ what the $\mathcal{DM}$ would be 'content' to have.

- Not all choice problems are always relevant. We thus allow the $\mathcal{DM}$'s behavior to be defined only on a restricted set D of subsets of X and refer to a pair (X, D) as *context*.

- An example: the $\mathcal{DM}$ is choosing whether to remain with the status quo s or choose an element in a set Y. Than $X = Y \cup \{s\}$ and D is the set of all subsets of X that contain s.

A rational choice function:

- A rational $\mathcal{DM}$ has a preference relation $\succsim$ on the set X and and given any choice problem $A \subseteq X$, $c(A)$ is $\succsim$-optimal.

- The induced choice function $c_{\succsim}$ is the function that assigns to every nonempty set $A \in D$ the $\succsim$-best element in A.

A choice function c can be rationalized if there is a preference relation $\succsim$ on X such that $c = c_{\succsim}$, that is

$$c(A) = c_{\succsim}(A) \text{ for any } A \subseteq X.$$

Rubinstein's condition α: a choice function c satisfies condition α if for any two problems $A, B \in D$, if $A \subset B$ and $c(B) \in A$, then $c(A) = c(B)$.

- The 2nd-best choice function violates condition α – $c(A)$ is the alternative from A that is the $\succsim$-maximal from among the non-maximal alternatives...

- It is a sufficient condition for c to be formulated as-if the $\mathcal{DM}$ is maximizing some preference relation $\succsim$.

I. Let c be a choice function defined on a domain containing at least all subsets of X of size of at most 3. If c satisfies condition α, then there is a preference $\succsim$ on X such that $c = c_{\succsim}$.

II. Let c be a choice function defined on a domain D satisfying that if two problems $A, B \in D$ then $A \cup B \in D$. If c satisfies condition α, then there is a preference $\succsim$ on X such that $c = c_{\succsim}$.

We will prove (II) and prove (I) when we consider choice correspondences (next week).

Proof: Define a binary relation as xRy if $x \neq y$ and there is a choice problem $A \in D$ such that $y \in A$ and $c(A) = x$.

– If xRy and yRz then there is $A \in D$ such that $y \in A$ and $c(A) = x$, and $B \in D$ such that $z \in B$ and $c(B) = y$.

– The set $A \cup B \in D$ and $c(A \cup B)$ is in either A or B and thus it is either x or y (by condition α).

– But if $c(A \cup B) = y$ then $c(A \cup B) = c(A) = y$ so $c(A \cup B) = x$ (again by condition α) and thus xRz.

! An acyclic and asymmetric relation R extends to a preference relation $\succsim$ (to be proved later) so $c(A) \succsim x$ for all $x \in A$ and thus $c(A) = c_{\succsim}(A)$.

Choice correspondences

The choice function $c_{\succsim}(A)$ requires that a single maximal element each choice problem – undefined for a preference relation with indifferences.

A choice correspondence c assigns to every nonempty $A \in D$ a nonempty subset of A so

$$\emptyset \neq c(A) \subseteq A.$$

$\Longrightarrow$ One (behavioral) interpretation of $c(A)$ is the set of all elements in A that are satisfactory in the sense of an "internal equilibrium."

$\Longrightarrow$ Another (behavioral) interpretation is that $c(A)$ is the set of all elements in A that are can be satisfactory under some possible circumstances not included in the description of the set A.

- Let (A, f) be an extended choice set where f is the 'frame' that accompanies the set A.

- Then $c(A, f)$ is the choice of the $\mathcal{DM}$ from A given frame f

$$C(A) = \{x | \, x = c(A, f) \text{ for some } f\}.$$

Two properties of choice correspondences:

(α) If $a \in A \subset B$ and $a \in c(B) \Longrightarrow a \in c(A)$.

(β) If $a, b \in A \subset B$, $a, b \in c(A)$ and $b \in c(B) \Longrightarrow a \in c(B)$.

For any domain D such that if $A, B \in D$ then $A \cap B \in D$, α and β are equivalent to the weak axiom of revealed preference (below).

$\Longrightarrow$ Verify this, but before provide examples of choice correspondences that satisfy one property but not the other...

The weak axiom of revealed preference (WA):

c satisfies the WA if whenever $x, y \in A \cap B$

$$x \in c(A) \text{ and } y \in c(B) \Longrightarrow x \in c(B).$$

In words, if x is revealed to be at least as good as y, then y is not revealed to be strictly better than x. Condition α for choice functions is replaced for correspondences by the WA.

Let c be a choice correspondence defined on a domain containing at least all subsets of X of size of at most 3. If c satisfies properties α and β, then there is a preference $\succsim$ on X such that $c = c_{\succsim}$.

<u>Proof</u>: Define $x \succsim y$ if $x \in c(\{x, y\})$. The relation $\succsim$ is a preference:

Reflexive	$c(\{x\}) = x$
Complete	$c(\{x, y\}) \neq \emptyset$

To see that is also transitive note that if $x \succsim y$ and $y \succsim z$ then $x \in c(\{x, y\})$ and $y \in c(\{y, z\})$.

– If $y \in c(\{x, y, z\})$ resp. $z \in c(\{x, y, z\})$ then

$$\begin{aligned} \alpha &\implies y \in c(\{x, y\}) \\ \beta &\implies x \in c(\{x, y, z\}) \end{aligned}$$

resp.

$$\begin{aligned} \alpha &\implies z \in c(\{y, z\}) \\ \beta &\implies y \in c(\{x, y, z\}) \end{aligned}$$

Thus, in all cases $x \in c(\{x, y, z\})$ so $\alpha \implies x \in c(\{x, z\})$ and therefore $x \succsim z$.

– It remains to be shown that $c(B) = c_{\succsim}(B)$:

$\rightarrow$ $x \in c(B)$ then (by α) $x \in c(\{x, y\})$ for every $y \in B$ so

$$x \succsim y \Longrightarrow x \in c_{\succsim}(B).$$

$\leftarrow$ $x \in c_{\succsim}(B)$ and $y \in c(B)$ then $x \in c(\{x, y\})$ since $x \succsim y$

$$\begin{aligned} \alpha &\Longrightarrow y \in c(\{x, y\}) \\ \beta &\Longrightarrow x \in c(B). \end{aligned}$$

Last word: procedural rationality

Consider the following Herbert Simon's so-called satisficing procedure:

- $v : X \rightarrow \mathbb{R}$ – the valuation of the elements in X where v^* is a threshold of satisfaction.

- Given a set $A \subseteq X$, $L(A, O)$ is the $\mathcal{DM}$'s list according to her/his ordering O.

$c(A)$ is the first element in $L(A, O)$ that has a v-value at least as large as v^* (and if there is no such element, the last element in $L(A, O)$ is chosen).

$\Longrightarrow$ The choice <u>function</u> $c(A)$ induced by this procedure satisfies condition α...