University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
Fall 2023

Classic demand theory
Part 1 (out of 2)
(Rubinstein Ch. 6 and Kreps Ch. 10-11)

Sep 21, 2023

Roadmap

We want to “complete” the development of the theory of the consumer (Rubinstein and Kreps do it in parallel with the theory of the firm):

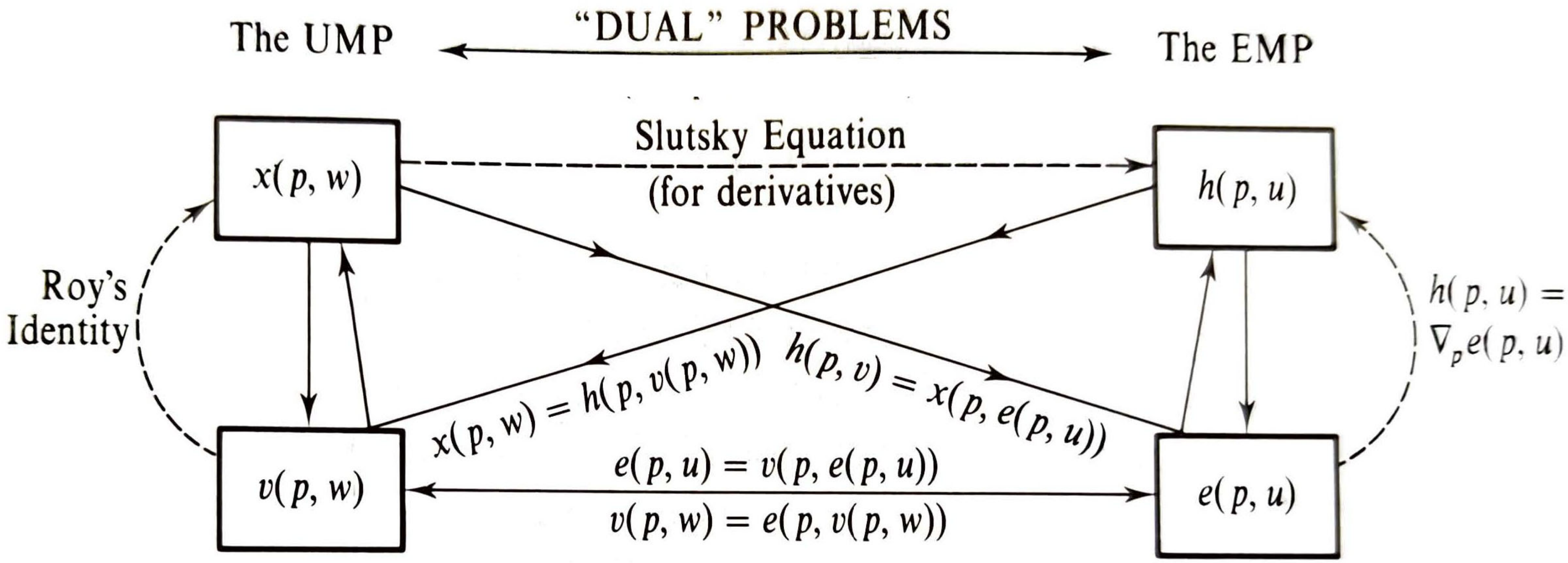
> When is a (parametric) family of demand functions represent the (Marshallian) demand of a utility-maximizing consumer?

This key question is answered (more or less) by the Integrability Theorem but “the path that takes us to this climax is long and winding...”

The key steps to reach this "climax" are Roy's identity and the Slutsky equation, which are two important "identities."

$\Longrightarrow$ Trying to isolate the substitution and income effects of a change in a price to

- the level of indirect utility (Roy's identity)
- the quantities consumed (the Slutsky equation).



The UMP
"DUAL" PROBLEMS
The EMP
Slutsky Equation
(for derivatives)
x(p, w)
h(p, u)
Roy's
Identity
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∇pe(p, u)
x(p, w) = h(p, v(p, w))
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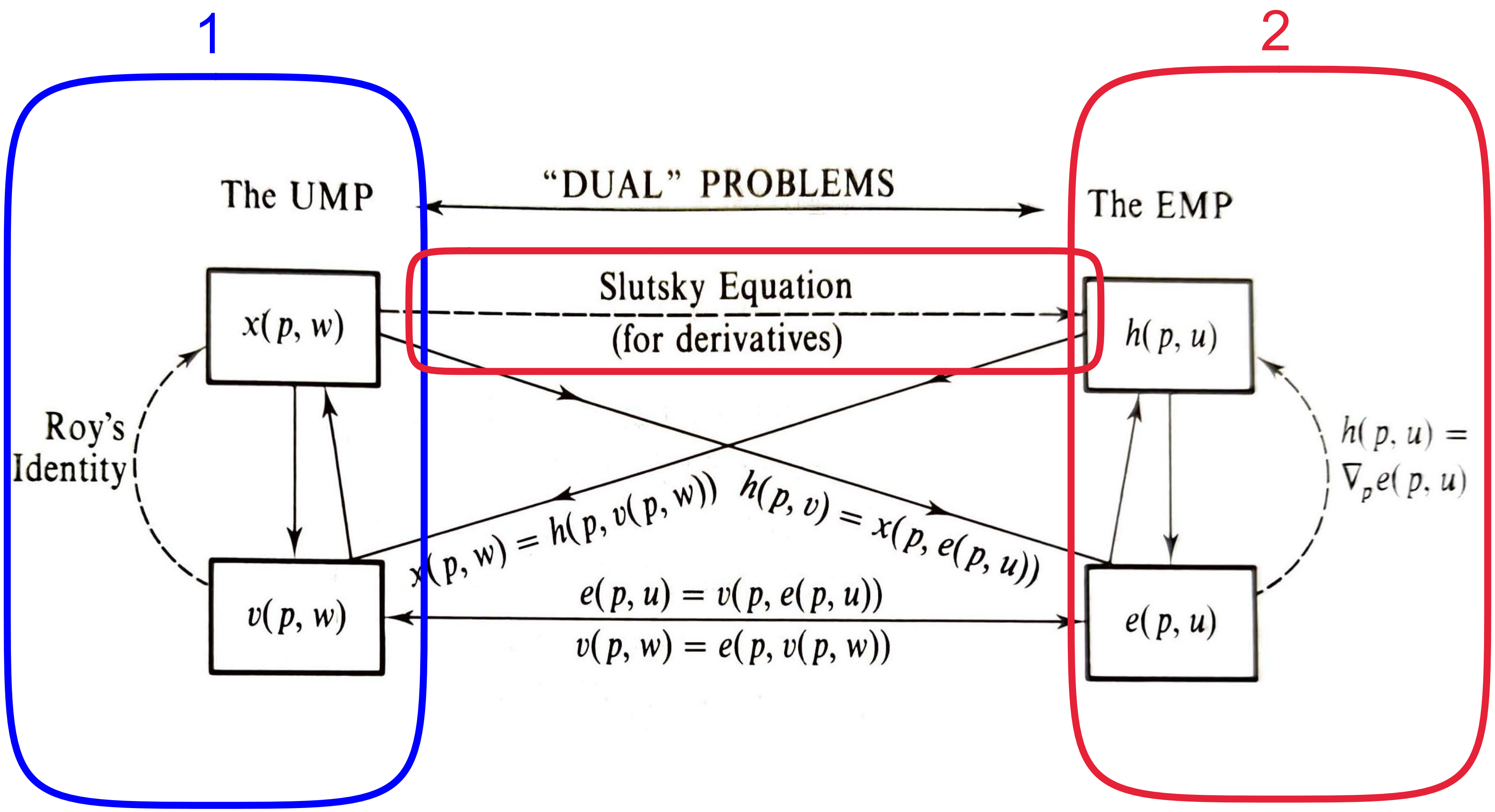
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The utility-maximization problem (UMP)

The budget set is given by

$$B = \{x : p \cdot x \leq \omega\}$$

where ω is the $\mathcal{DM}$'s income/wealth and p is the vector of commodity prices. The UMP can then be written as:

$$\begin{aligned} &\max_x u(x) \\ &\text{subject to } p \cdot x \leq \omega. \end{aligned}$$

The value of x that solves this problem is the demanded bundle $x^* = x(p, \omega)$, which is not necessarily unique (requires strict convexity of $\succsim$).

By making a few 'regularity' assumptions on $\succsim$, we can say more about the solution to the UMP x^*.

Budget balancedness (Walras' Law) allows us to restate the UMP as follows:

$$\begin{aligned} v(p,\omega) &= \max_x u(x) \\ &\text{subject to } p \cdot x = \omega. \end{aligned} \tag{1}$$

where $v(p,\omega)$ is the indirect utility function, which gives us the maximum utility achievable at prices p and income ω.

The standard properties of the indirect utility function $v(p, \omega)$:

1. non-increasing in p_k and non-decreasing in ω.

2. homogeneous of degree 0 in (p, ω).

3. quasi-convex in p , that is the set

$$\{p : v(p, \omega) \leq u\}$$

   is convex for all u.

4. continuous at all $p >> 0$ and $\omega > 0$.

(3) $\underline{v(p, \omega) \text{ quasi-convex in } p}$:

- Suppose p and $p' \neq p$ are such that $v(p, \omega) \leq u$ and $v(p', \omega) \leq u$, let $p'' = \alpha p' + (1 - \alpha)p'$ for $\alpha \in [0, 1]$.

- Define B, B', and B'' accordingly. It is sufficient to show that $B'' \subset B \cup B'$ (so any $x \in B''$ must be also in either B or B').

- Assume not:

$$\begin{aligned} x &\notin B \Longrightarrow \alpha px > \alpha\omega \\ x &\notin B' \Longrightarrow (1 - \alpha)p'x > (1 - \alpha)\omega. \end{aligned}$$

So $\alpha px + (1 - \alpha)p'x > \omega$, which contradicts that $x \in B''$.$\blacksquare$

The expenditure-minimization problem (EMP)

(Kreps Ch. 9 for a profit-maximizing firm)

If $\succsim$ satisfy the local non satiation assumption then $v(p, \omega)$ is strictly increasing in ω so we can invert it and solve for ω as a function of the level of utility.

The expenditure function $e(p, u)$ – the inverse of $v(p, \omega)$ – indicates the minimal income ω needed to achieve utility level u at prices p:

$$\begin{aligned} e(p, u) &= \min_x p \cdot x \\ \text{subject to } & u(x) \geq u. \end{aligned} \tag{2}$$

The standard properties of the expenditure function $e(p, u)$:

1. non-decreasing in p.

2. homogeneous of degree $\mathbf{1}$ in p.

3. concave in p.

4. continuous in p for all $p >> \mathbf{0}$.

The expenditure function is (completely) analogous to the cost function in the theory of the firm.

Let $h(p, u)$ be the expenditure-minimizing bundle so

$$h_k(p, u) = \frac{\partial e(p, u)}{\partial p_k} \text{ for } k = 1, ..., K$$

(assuming differentiability). The function $h(p, u)$ is called the Hicksian (or compensated) demand function.

When we want to emphasize the difference between the Hicksian demand function $h(p, u)$ and the 'usual' demand function $x(p, \omega)$, we refer to the latter as the Marshallian demand function.

The “dual” problems: the relationships between UMP and EMP

The (simple) observation that the solution x^* to the UMP (1) is the same as the solution to the EMP (2) leads to four important identities:

(1A) The maximum utility achievable from income $e(p, u)$ is u

$$v(p, e(p, u)) \equiv u.$$

(1B) The minimum expenditure necessary to achieve utility $v(p, \omega)$ is ω

$$e(p, v(p, \omega) \equiv \omega.$$

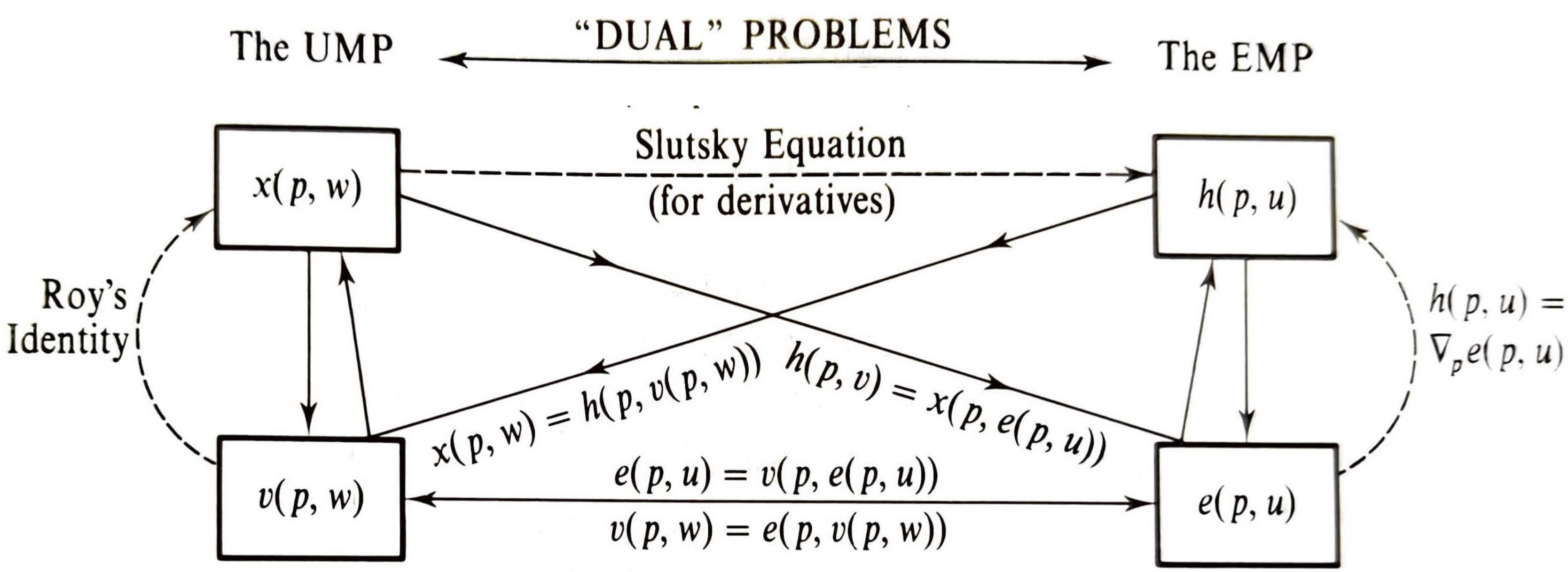
(2A) The Marshallian demand at income ω is the same as the Hicksian demand at utility $v(p, \omega)$

$$x(p, \omega) \equiv h(p, v(p, \omega)).$$

(2B) The Hicksian demand at utility u is the same as the Marshallian demand at income $e(p, u)$

$$h(p, u) \equiv x(p, e(p, u)).$$

! The last identity is (perhaps) the most important (for empirical work) since it ties together the observable Marshallian demand with the unobservable Hicksian demand.



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The indirect utility function and Roy's identity (a non-standard discussion)

Consider a consumer who is choosing among budget sets. We will formulate (and study) the "indirect" preferences of the consumer on budget sets.

More broadly, we can think of a $\mathcal{DM}$ choosing between choice sets where X is the set of alternatives and D the set of choice problems (non-empty subsets of X).

– The "indirect" preference relation: $\succsim^*$ is the indirect preference relation induced by $\succsim$ if

$$C_{\succsim}(A) \succsim C_{\succsim}(B) \Longrightarrow A \succsim^* B \text{ for any } A, B \in D.$$

(i) $\succsim^*$ is a preference relation, and if u represents $\succsim$ and $C_{\succsim}$ is well defined, then

$$v(A) = u(C_{\succsim}(A))$$

represents $\succsim^*$ so v is the indirect utility function.

(ii) Depending on the set of choice problems D, the choice function $C_{\succsim}$

– can be reconstructed from the indirect preferences $\succsim^*$, e.g. if $a \in A$ and

$$A \succ^* A - \{a\} \Longrightarrow C_{\succsim}(A) = a.$$

Gul and Pesendorfer's (2001) temptation and self-control:

– A "standard" $\mathcal{DM}$ will always prefer a bigger choice set to a smaller choice set (in the subset sense):

$$B \subset A \Longrightarrow A \succsim^* B.$$

Otherwise, $\succ^*$ exhibits a preference for commitment (at A). $\succ^*$ has a preference for commitment if it has a preference for commitment at some choice set A.

– Gul and Pesendorfer's (2001): so-called set betweenness

$$A \succsim^* B \Longrightarrow A \succsim^* A \cup B \succsim^* B$$

permits a preference for commitment.

Now, let's get back to a consumer who is choosing among budget sets (characterized by the $K+1$ parameters (p,ω)).

If $\succsim$ is well-behaved (satisfies monotonicity, continuity and convexity) and the demand $x(p,\omega)$ is always well-defined. then the indirect preferences $\succsim^*$is defined by

$$(p,\omega) \succsim^* (p',\omega') \text{ if } x(p,\omega) \succsim x(p',\omega').$$

The properties of indirect preferences $\succsim^*$: Same as above for the indirect utility function $v(p, \omega)$ as they follows directly from the properties of $x(p, \omega)$.

(3) The 'concavity' of $\succsim^*$($v(p, \omega)$ to be quasi-convex in p): For any $\alpha \in$ [0, 1]

$$(p, \omega) \succsim^* (p', \omega') \Longrightarrow (p, \omega) \succsim^* (p'', \omega'').$$

Let $z = x(p'', \omega'')$ so

$$p''z \leq \omega'' \Longrightarrow pz \leq \omega \text{ or } p'z \leq \omega' \Longrightarrow$$
$$\Longrightarrow x(p, \omega) \succsim z \text{ or } x(p', \omega') \succsim z.$$

Because $x(p, \omega) \succsim x(p', \omega')$, we conclude that $x(p, \omega) \succsim z$.■

Roy's identity

A method for recovering $x(p, \omega)$ from $\succsim^*$

– When $K = 1$, each $\succsim^*$-indifference curve is a ray. If $\succsim$ are well-behaved (monotonic), then the slope of the indifference curve through (p_1, ω) is $\dfrac{\omega}{p_1}$, which is $x_1(p_1, \omega)$.

– For any K-commodity space, the set (hyperplane)

$$H = \{(p, \omega) : p \cdot x(p^*, \omega^*) = \omega\}$$

is tangent to the $\succsim^*$-indifference curve through (p^*, ω^*), and if the tangent is unique, then knowing that tangent enables us to recover $x(p^*, \omega^*)$.

If $\succsim$ satisfies monotonicity then $(p^*, \omega^*) \in H$ and $x(p^*, \omega^*) \in B(p, \omega)$ for any $(p, \omega) \in H$. Therefore,

$$x(p, \omega) \succsim x(p^*, \omega^*) \Longrightarrow (p, \omega) \succsim^* (p^*, \omega^*).$$

– Roy's identity: If $\succsim^*$are represented by a differentiable indirect utility function v, then

$$\begin{aligned} x_k(p^*, \omega^*) &= -\frac{\nabla_p v(p^*, \omega^*)}{\nabla_\omega v(p^*, \omega^*)} \\ &= -\frac{\partial v / \partial p_k(p^*, \omega^*)}{\partial v / \partial \omega(p^*, \omega^*)} \text{ for all } k = 1, ..., K. \end{aligned}$$

Proof (envelope theorem argument):

Applied to the UMP the envelope theorem tells us:

$$\partial v/\partial p_k(p^*, \omega^*) = -\lambda x_k(p^*, \omega^*)$$

and

$$\partial v/\partial \omega(p^*, \omega^*) = -\lambda$$

where λ is the Lagrange multiplier, which yield the result.■

Whoa. Dude, Mr. Turtle is my father. Name's Crush.

Rubinstein's "dual" turtle
(a preface to the "dual" consumer)

Consider the following two statements about Crush (the sea turtle in Finding Nemo):

(1) The maximal distance Crush can swim in 1 hour is 20 miles.
(2) The minimal time is takes Crush to swim 20 miles is 1 hour.

– (1) $\Rightarrow$ (2) if Crush swims a positive distance in any period of time.

– (1) $\Leftarrow$ (2) if Crush cannot "jump" a positive distance in zero time.

The maximal distance Crush can swim in time t $M(t)$ must be strictly increasing and continuous.

Last word (for today): 4 (relevant) quotes by Crush:

- You, mini-man! Taking on the jellies. You got serious thrill issues, dude. Awesome!
- I saw the whole thing, dude. First you were all, like, whoa! And then we were all, like, whoa! Then you were, like, whoa...
- When the little dudes are just eggs we leave them on a beach to hatch, and coo-coo-cachoo, they find their way back to the big ol' blue...
- Oh, man, no hurling on the shell, dude. Okay? I just waxed it...