University of California – Berkeley
Department of Economics
ECON 201A Economic Theory
Choice Theory
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The basics of consumer demand
(Kreps Ch. 3 and Rubinstein Ch. 4-5)

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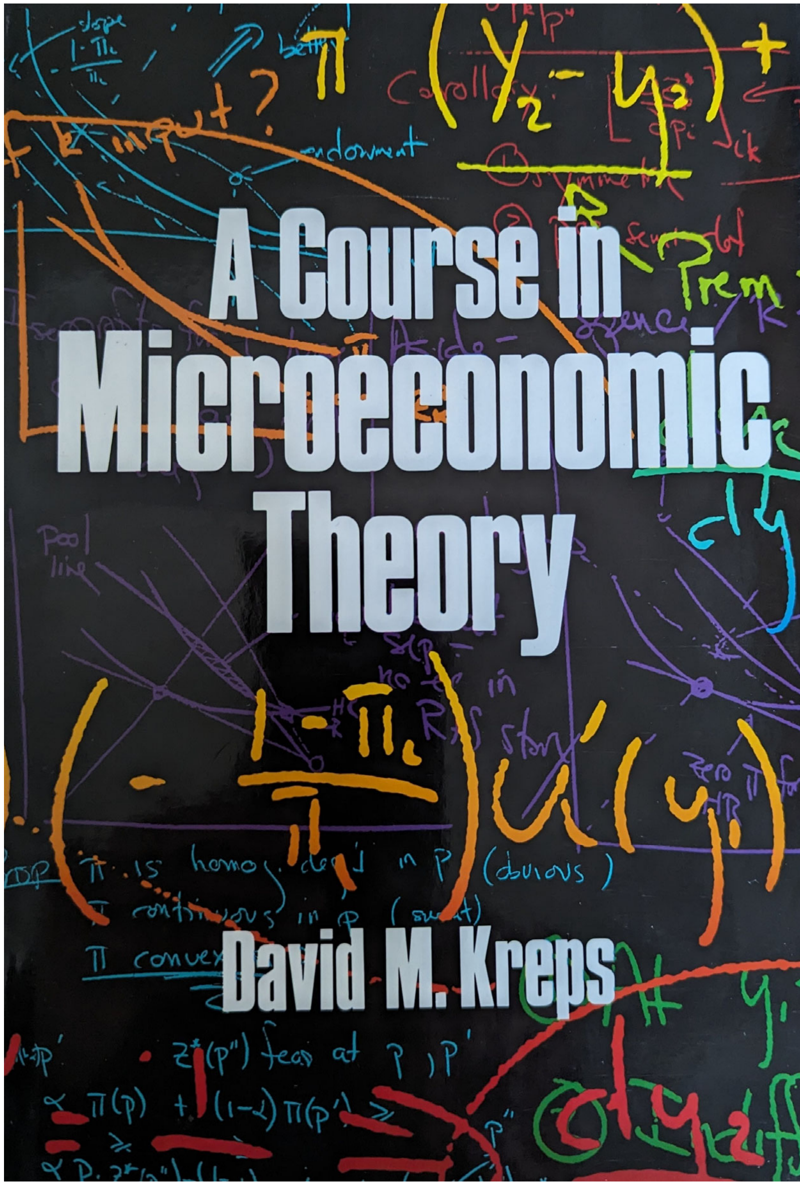

contents

Are utility functions differentiable?

"The axioms of the theory must be formulated in terms of observable choices made by a consumer among commodity vectors..." (Debreu, 1972)

"Nothing in our axioms will guarantee this nor can new axioms be introduced to do so without making patently unrealistic assumptions..." (Deaton and Muellbauer, 1980)

! Smooth rationalization: necessary + sufficient conditions for rationalizing price-quantity data by a well-behaved and <u>differentiable</u> utility function.

Differentiable preferences, whatever it means...

Consider a vector of (subjective) values $v(x) \in \mathbb{R}_+^K$ for the K commodities and a feasible the direction $x + \varepsilon d \in X$ from x for small enough $\varepsilon > 0$.

d is considered to be an improvement by the $\mathcal{DM}$ if and only if

$$d \cdot v(x) > 0.$$

Given $v(x) : X \to \mathbb{R}_+^K$, let

$$D_v(x) = \{d : d \cdot v(x) > 0\}$$

be the set of directions that are improvements relative to x.

$d \in \mathbb{R}^K$ is an <u>improvement direction</u> at x if there is $\lambda^* > 0$ such that λd is an improvement

$$x + \lambda d \succ x$$

for any $\lambda \leq \lambda^*$. Let $D_{\succsim}(x)$ be the set of all improvement directions at x:

Any improvement is an improvement direction if

- $\succsim$ are strictly convex.
- $\succsim$ are convex, strongly monotonicity, and continuous (verify this!).

A ‘differentiable’ $\succsim$ (a nonconventional definition): $\succsim$ is differentiable if there exists a function $v(x) : X \rightarrow \mathbb{R}_+^K$ such that

$$D_{\succsim}(x) = D_v(x) \text{ for all } x \in X.$$

$\succsim$ represented by

- $\alpha x_1 + \beta x_2$ for $\alpha, \beta > 0$ are differentiable: $v(x) = (\alpha, \beta)$.

- $\min\{\alpha x_1, \beta x_2\}$ are differentiable where $\alpha x_1 \neq \beta x_2$:

$$v(x) = \begin{cases} (1,0) & \text{if } \alpha x_1 < \beta x_2 \\ (0,1) & \text{otherwise.} \end{cases}$$

Any (monotonic + convex) $\succsim$ that can be represented by a (strongly monotonic + quasi-concave) and differentiable u is differentiable.

Proof:

Let $\nabla u(x)$ be the vector of partial derivatives $\dfrac{du}{dx_k}(x)$ (the gradient) so the rate of change of u when moving from x in any direction d is

$$d \cdot \nabla u(x).$$

Let $v(x) = \nabla u(x)$. We want to show that $D_{\succsim}(x) = D_v(x)$, that is

$$D_{\succsim}(x) \subseteq D_v(x) \text{ and } D_{\succsim}(x) \supseteq D_v(x).$$

- $D_{\succsim}(x) \supseteq D_v(x)$: Assume $d \in D_v(x)$. Since u is differentiable, for any small enough ε

$$d \cdot v(x) > 0 \Longrightarrow u(x + \epsilon d) > u(x)$$

  and thus $d \in D_{\succsim}(x)$.

- $D_{\succsim}(x) \subseteq D_v(x)$: Assume $d \in D_{\succsim}(x)$ such that $d \cdot v(x) \leq 0$ (towards contradiction).

(i) Continuity of $\succsim$: For any $d' \neq d$ and $d'(k) \leq d(k)$ for all k

$$x + d \succ x \Longrightarrow x + d' \succ x.$$

(ii) Convexity and strong monotonicity of $\succsim$ (follows from the quasi-concavity and strong monotonicity of u):

$$d' \in D_{\succsim}(x).$$

(iii) But $d' \cdot v(x) < 0$ (since that $d \cdot v(x) \leq 0$) and thus (by the differentiability of u)

$$u(x + \delta d') < u(x) \text{ for any small enough } \delta.$$

! In empirical work, u-functions are differentiable, monotonic and quasi-concave. $\succsim$ represented by such u are differentiable.

Budget sets

We continue to have $X = \mathbb{R}_+^K$. The budget set is given by

$$B = \{x : p \cdot x \leq \omega\}$$

where ω is the $\mathcal{DM}$'s wealth and p is the vector of commodity prices. WLOG, ω is normalized to 1.

Trivially, $B(p)$ is compact:

– closed: defined by $\leq$.

– bounded: for any $x \in B(p)$, $0 \leq x_k \leq \dfrac{1}{p_k}$ for all k.

– convex: if $x, x' \in B(p)$ then $p \cdot x \leq 1$ and $p \cdot x' \leq 1$ so for any $\alpha \in [0, 1]$

$$\begin{aligned} 1 &\geq \alpha p x + (1-\alpha) p x' \\ &= p[\alpha x + (1-\alpha) x']. \end{aligned}$$

Since $x_k, x'_k \geq 0$ for all k

$$\alpha x + (1-\alpha) x' \in B(p).$$

The problem of a $\mathcal{DM}$ called the consumer...

The $\mathcal{DM}$'s problem is finding the $\succsim$-best bundle $x \in B(p)$.

$\succsim$ is continuous:

If $\succsim$ is continuous, then all such problems have a solution.

– Proof 1: If $\succsim$ is continuous, then it admits a continuous u-representation so finding the $\succsim$-best bundle is equivalent to

$$x \in \arg\max_{x \in B(p)} u(x).$$

Since the $B(p)$ is compact and u is continuous, the problem has a solution (Weierstrass Theorem).

– <u>Proof 2</u>: For any $x \in B(p)$, define its inferior set

$$x_{\prec} = \{y \in X : y \prec x\}$$

which is open (because $\succsim$ is continuous), and assume there is no $\succsim$-best bundle $x \in B(p)$.

Thus, any $z \in B(p)$ is also $z \in x_{\prec}$ (for some $x_{\prec}$) so the collection of all $x_{\prec}$ 'covers' $B(p)$.

But collection of open sets that 'covers' a compact set has a finite sub-collection (Heine-Borel Theorem).

The $\succsim$-best bundle is this sub-collection is a $\succsim$-best in $B(p)$.

$\succsim$ <u>is convex</u>:

- If $\succsim$ is <u>convex</u>, then the set $\succsim$-best bundles $x \in B(p)$ (or any other convex set) is convex.

  Assume x and x' are $\succsim$-best bundles. Since $B(p)$ is convex

  $$x, x' \in B(p) \Longrightarrow \alpha x + (1 - \alpha)x' \in B(p)$$

  and since $\succsim$ is convex

  $$\alpha x + (1 - \alpha)x' \succsim x \succsim z$$

  for all $z \in B(p)$ and thus $\alpha x + (1 - \alpha)x'$ is also a $\succsim$-best bundle.

– If $\succsim$ is strictly convex, then the set $\succsim$-best bundles $x \in B(p)$ is (at most) a singleton.

If x and $x' \neq x$ are $\succsim$-best bundles then $x \sim x'$. Again, since $B(p)$ is convex

$$x, x' \in B(p) \Longrightarrow \alpha x + (1 - \alpha)x' \in B(p)$$

and since $\succsim$ is strictly convex

$$\alpha x + (1 - \alpha)x' \succ x,$$

a contradiction.

$\succsim$ <u>is ‘differentiable’</u>:

Assume that $\succsim$ is differentiable and denote the vector of “subjective value numbers” at x^* (as defined above) by

$$v(x^*) = (v_1(x^*), ..., v_K(x^*)).$$

If $x^* \in B(p)$ is a $\succsim$-best bundle then

$$\frac{v_k(x^*)}{v_j(x^*)} = \frac{p_k}{p_j} \text{ for any } x_k^*, x_j^* > 0.$$

These are the “classic” necessary conditions by taking $v_k(x^*) = \dfrac{\partial u}{\partial x_k}(x^*)$.

If x^* is a $\succsim$-optimal bundle and k is a consumed commodity $x_k^* > 0$, then it must be that

$$\frac{v_k(x^*)}{p_k} \geq \frac{v_j(x^*)}{p_j} \text{ for any } j \neq k. \qquad (*)$$

In words, the "value per dollar" at a bundle of a commodity that is positively consumed is as large as the "value per dollar" of any other commodity.

Assume that $x^* \in B(p)$ is a $\succsim$-best bundle but that the inequality above is reversed so

$$v_j(x^*)\frac{p_k}{p_j} - v_k(x^*) > 0$$

to get the "standard" figure, contradicting the assumption that $x^*$is a $\succsim$-optimal bundle in $B(p)$.

$\succsim$ is strongly monotonic, convex, continuous, and differentiable:

If $\succsim$ is strongly monotonic, convex, continuous, and differentiable and if $p \cdot x^* = 1$ (budget balancedness) and $(*)$ is satisfied then x^* is a $\succsim$-optimal bundle.

Proof: If $x^* \in B(p)$ is not a $\succsim$-optimal bundle then there is $y \in B(p)$ such that $y \succ x^*$ and so $(y - x^*)$ is an improvement direction (as defined above).

Let $\mu = \dfrac{v_k(x^*)}{p_k}$ for all k such that $x_k^* > 0$ and note that

$$\begin{aligned} 0 &\geq p(y - x^*) \\ &= \sum p_k(y_k - x_k^*) \\ &\geq \sum \frac{v_k(x^*)}{\mu}(y_k - x_k^*) \end{aligned}$$

where the 1st $\geq$ is because $p \cdot x^* = 1$ and the 2nd $\geq$ is because $p_k \geq v_k(x^*)/\mu$ when $x_k^* = 0$ and $(y_k - x_k^*) \geq 0$. Thus $v_k(x^*) \cdot (y_k - x_k^*) \leq 0$, a contradiction with $(y - x^*)$ being an improvement direction.

The demand function

A demand function is a choice function $x(p)$ that assigns to every vector of prices p a unique bundle $x \in B(p)$.

(i) $x(p)$ is homogeneous of degree zero: since $B(\lambda p, \lambda\omega) = B(p, \omega)$

$$x(\lambda p, \lambda\omega) = x(p, \omega).$$

(ii) If $\succsim$ is a continuous preference relation, then the induced demand function is continuous in prices (and in wealth).

! Since continuous $\succsim$ admits a continuous u-representation, a standard "maximum theorem" cab be applied but there is also a (simple) proof that does not use u-maximization.

Walras' Law

Walras' Law: if $\succsim$ is monotonic, then any solution x to the consumer problem $B(p)$ satisfies budget balancedness, that is $p \cdot x(p) = 1$.

Otherwise, there is an $\varepsilon > 0$ such that $(x_1+\varepsilon, ..., x_K+\varepsilon) \in B(p)$ and by monotonicity of $\succsim$

$$(x_1 + \varepsilon, ..., x_K + \varepsilon) \succ x,$$

contradicting that x is optimal in $B(p)$.

Rationalizable demand functions

1. $\succsim$ <u>fully rationalize</u> the demand function x if for any p the bundle $x(p)$ is the <u>unique</u> $\succsim$-maximal bundle within $B(p)$.

2. $\succsim$ is "being rationalizable" if for any p the bundle $x(p)$ is a $\succsim$-maximal bundle (though not necessarily unique) within $B(p)$.

   An "empty" definition $\Longrightarrow$ any $x(p)$ is consistent with $\succsim$-maximal for "total indifference."

3. A monotonic $\succsim$ <u>rationalize</u> the demand function x if for any p the bundle $x(p)$ is a $\succsim$-maximal bundle within $B(p)$.

To illustrate, the demand function

$$x(p) = \begin{cases} (\frac{1}{p_1}, 0) & \text{if} \quad p_1 \geq p_2 \\ (0, \frac{1}{p_2}) & \text{if} \quad p_1 < p_2 \end{cases}$$

is not rationalizable.

Next: We look for general conditions that will guarantee that a demand function x can be fully rationalized (i.e. $x(p)$ is the unique $\succsim$-maximal bundle within $B(p)$ for some $\succsim$).

Condition α: necessary + sufficient for a choice function to be derived from some $\succsim$ under certain assumptions about the choice domain X.

- α do not apply to budget sets: $B(p)$ is infinite and $B \cup B'$ is not a budget set.

- Instead, we use the "revealed preference" induced from a demand function $x(p)$.

The weak and strong axioms of revealed preference

A demand function $x(p)$ satisfies the WA if $x(p) \neq x(p')$

$$p \cdot x(p') \leq 1 \Longrightarrow p' \cdot x(p) > 1.$$

– A necessary and sufficient condition for $x(p)$ (which satisfies Walras' law and homogeneity of degree zero) to be rationalizable is the SA.

– A classic example with $K = 3$ is presented in Hicks (1956). With $K = 2$, any violation of the SA is also a violation of the WA...