Economics 201B
Economic Theory
(Spring 2022)
Bargaining

Topics: the axiomatic approach (OR 15) and the strategic approach (OR 7).

The axiomatic approach (OR 15)

Nash's (1950) work is the starting point for formal bargaining theory.

The bargaining problem consists of

- a set of utility pairs that can be derived from possible agreements, and
- a pair of utilities which is designated to be a disagreement point.

The bargaining solution is a function that assigns a unique outcome to every bargaining problem.

Nash's bargaining solution is the first solution that

- satisfies four plausible conditions, and
- has a simple functional form, which make it convenient to apply.

A bargaining situation

A bargaining situation is a tuple $\langle N, A, D, (\succsim_i)\rangle$ where

- N is a set of players or bargainers ($N = \{1, 2\}$),
- A is a set of agreements/outcomes,
- D is a disagreement outcome, and
- $\succsim_i$ is a preference ordering over the set of lotteries over $A \cup \{D\}$.

The objects N, A, D and $\succsim_i$ for $i = \{1, 2\}$ define a bargaining situation.

$\succsim_1$ and $\succsim_2$ satisfy the assumption of vNM so for each i there is a utility function $u_i : A \cup \{D\} \rightarrow \mathbb{R}$.

$\langle S, d \rangle$ is the primitive of Nash's bargaining problem where

- $S = (u_1(a), u_2(a))$ for $a \in A$ the set of all utility pairs, and
- $d = (u_1(D), u_2(D))$.

A bargaining problem is a pair $\langle S, d \rangle$ where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$ and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems $\langle S, d \rangle$ is denoted by B.

A bargaining solution is a function $f : B \rightarrow \mathbb{R}^2$ such that f assigns to each bargaining problem $\langle S, d \rangle \in B$ a unique element in S.

The definitions of the bargaining problem and solution have few restrictions (the convexity assumption on S is more technical):

- bargaining situations that induce the same pair $\langle S, d \rangle$ are treated identically,

- the utilities obtainable in the outcome of bargaining are limited since S is bounded,

- players can agree to disagree since $d \in S$ and there is an agreement preferred by both players to the disagreement outcome.

Nash's axioms

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely - Nash 1953 -

Does not capture the details of a specific bargaining problem (e.g. alternating or simultaneous offers).

Rather, the approach consists of the following four axioms: invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and (weak) Pareto efficiency.

Invariance to equivalent utility representations (INV)

$\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ if

$$d'_i = \alpha_i d_i + \beta_i$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Note that if $\alpha_i > 0$ for $i = 1, 2$ then $\langle S', d' \rangle$ is itself a bargaining problem.

If $\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ where $\alpha_i > 0$ for each i, then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$$

for $i = 1, 2$. Hence, $\langle S', d' \rangle$ and $\langle S, d \rangle$ represent the same situation.

INV requires that the utility outcome of the bargaining problem co-vary with representation of preferences.

The physical outcome predicted by the bargaining solution is the same for $\langle S', d' \rangle$ and $\langle S, d \rangle$.

A corollary of INV is that we can restrict attention to $\langle S, d \rangle$ such that

$S \subset \mathbb{R}^2_+$,

$S \cap \mathbb{R}^2_{++} \neq \emptyset$, and

$d = (0, 0) \in S$ (reservation utilities).

Symmetry (SYM)

A bargaining problem $\langle S, d \rangle$ is symmetric if $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$. If the bargaining problem $\langle S, d \rangle$ is symmetric then

$$f_1(S, d) = f_2(S, d)$$

Nash does not describe differences between the players. All asymmetries (in the bargaining abilities) must be captured by $\langle S, d \rangle$.

Hence, if players are the same the bargaining solution must assign the same utility to each player.

Independence of irrelevant alternatives (IIA)

If $\langle S, d \rangle$ and $\langle T, d \rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$ then

$$f(S, d) = f(T, d)$$

If T is available and players agree on $s \in S \subset T$ then they agree on the same s if only S is available.

IIA excludes situations in which the fact that a certain agreement is available influences the outcome.

Weak Pareto efficiency (WPO)

If $\langle S, d \rangle$ is a bargaining problem where $s \in S$ and $t \in S$, and $t_i > s_i$ for $i = 1, 2$ then $f(S, d) \neq s$.

In words, players never agree on an outcome s when there is an outcome t in which both are better off.

Hence, players never disagree since by assumption there is an outcome s such that $s_i > d_i$ for each i.

$\underline{SYM \text{ and } WPO}$

restrict the solution on single bargaining problems.

$\underline{INV \text{ and } IIA}$

requires the solution to exhibit some consistency across bargaining problems.

Nash 1953: there is precisely one bargaining solution, denoted by $f^N(S,d)$, satisfying SYM, WPO, INV and IIA.

Nash's solution

The unique bargaining solution $f^N : B \to \mathbb{R}^2$ satisfying SYM, WPO, INV and IIA is given by

$$f^N(S,d) = \underset{(d_1,d_2)\le(s_1,s_2)\in S}{\arg\max} (s_1 - d_1)(s_2 - d_2)$$

and since we normalize $(d_1, d_2) = (0, 0)$

$$f^N(S,0) = \underset{(s_1,s_2)\in S}{\arg\max} s_1 s_2$$

The solution is the utility pair that maximizes the product of the players' utilities.

Proof

Pick a compact and convex set $S \subset \mathbb{R}^2_+$ where $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

Step 1: f^N is well defined.

- Existence: the set S is compact and the function $f = s_1 s_2$ is continuous.
- Uniqueness: f is strictly quasi-conacave on S and the set S is convex.

Step 2: f^N is the only solution that satisfies SYM, WPO, INV and IIA.

Suppose there is another solution f that satisfies SYM, WPO, INV and IIA.

Let

$$S' = \{(\frac{s_1}{f_1^N(S)}, \frac{s_2}{f_2^N(S)}) : (s_1, s_2) \in S\}$$

and note that $s'_1 s'_2 \leq 1$ for any $s' \in S'$, and thus $f^N(S', 0) = (1, 1)$.

Since S' is bounded we can construct a set T that is symmetric about the 45° line and contains S'

$$T = \{(a, b) : a + b \leq 2\}$$

By WPO and SYM we have $f(T, 0) = (1, 1)$, and by IIA we have $f(S', 0) = f(T, 0) = (1, 1)$.

By INV we have that $f(S', 0) = f^N(S', 0)$ if and only if $f(S, 0) = f^N(S, 0)$ which completes the proof.

Is any axiom superfluous?

$\underline{INV}$

The bargaining solution given by the maximizer of

$$g(s_1, s_2) = \sqrt{s_1} + \sqrt{s_2}$$

over $\langle S, 0 \rangle$ where $S := co\{(0,0),(1,0),(0,2)\}$.

This solution satisfies WPO, SYM and IIA (maximizer of an increasing function). The maximizer of g for this problem is $(1/3, 4/3)$ while $f^N = (1/2, 1)$.

$\underline{SYM}$

The family of solutions $\{f^\alpha\}_{\alpha\in(0,1)}$ over $\langle S,0\rangle$ where

$$f^\alpha(S,d) = \underset{(d_1,d_2)\le(s_1,s_2)\in S}{\arg\max}(s_1-d_1)^\alpha(s_2-d_2)^{1-\alpha}$$

is called the asymmetric Nash solution.

Any f^α satisfies INV, IIA and WPO by the same arguments used for f^N.

For $\langle S,0\rangle$ where $S := co\{(0,0),(1,0),(0,1)\}$ we have $f^\alpha(S,0) = (\alpha, 1-\alpha)$ which is different from f^N for any $\alpha \neq 1/2$.

<u>WPO</u>

Consider the solution f^d given by $f^d(S, d) = d$ which is different from f^N. f^d satisfies INV, SYM and IIA.

WPO in the Nash solution can be replaced with <u>strict individual rationality</u> (SIR) $f(S, d) >> d$.

An application - risk aversion

Dividing a dollar: the role of risk aversion: Suppose that

$$A = \{(a_1, a_2) \in \mathbb{R}^2_+ : a_1 + a_2 \leq 1\}$$

(all possible divisions), $D = (0, 0)$ and for all $a, b \in A$ $a \succsim_i b$ if and only if $a_i \geq b_i$.

Player i's preferences over $A \cup D$ can be represented by $u_i : [0, 1] \rightarrow \mathbb{R}$ where each u_i is concave and (WLOG) $u_i(0) = 0$.

Then,

$$S = \{(s_1, s_2) \in \mathbb{R}^2_+ : (s_1, s_2) = (u_1(a_1), u_2(a_2))\}$$

for some $(a_1, a_2) \in A$ is compact and convex and

$$d = (u_1(0), u_2(0)) = (0, 0) \in S.$$

First, note that when $u_1(a) = u_2(a)$ for all $a \in (0, 1]$ then $\langle S, d \rangle$ is symmetric so by SYM and WPO the Nash solution is $(u(1/2), u(1/2))$.

Now, suppose that $v_1 = u_1$ and $v_2 = h \circ u_2$ where $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is increasing and concave and $h(0) = 0$ (player 2 is more risk averse).

Let $\langle S', d' \rangle$ be bargaining problem when the preferences of the players are represented by v_1 and v_2.

Let z_u be the solution of

$$\max_{0 \leq z \leq 1} u_1(z) u_2(1 - z),$$

and z_v the corresponding solution when $u_i = v_i$ for $i = 1, 2$.

Then,

$$f^N(S,d) = (u_1(z_u), u_2(1-z_u)) \text{ and } f^N(S',d') = (v_1(z_v), v_2(1-z_v)).$$

If u_i for $i = 1, 2$ and h are differentiable then z_u and z_v are, in respect, the solutions of

$$\frac{u_1'(z)}{u_1(z)} = \frac{u_2'(1-z)}{u_2(1-z)}, \tag{1}$$

and

$$\frac{u_1'(z)}{u_1(z)} = \frac{h'(u_2(1-z))u_2'(1-z)}{h(u_2(1-z))}. \tag{2}$$

Since h is increasing and concave and $h(0) = 0$ we have

$$h'(t) \leq \frac{h(t)}{t}$$

for all t, so the RHS of (1) is at least as the RHS of (2) and thus $z_u \leq z_v$. Thus, if player 2 becomes more risk-averse, then f_1^N increases and f_2^N decreases.

If player 2's marginal utility declines more rapidly than that of player 1, then player 1's share exceeds $1/2$.

Monotonicity

Individual monotonicity (INM)

Let $\bar{s}_i$ be the maximum utility player i gets in $\{s \in S : s \geq d\}$.

(i) For any $\langle S, d \rangle$ and $\langle T, d \rangle$ with $S \subset T$ and $\bar{s}_i = \bar{t}_i$ for $i = 1, 2$, we have

$$f_i(S, d) \leq f_i(T, d)$$

for $i = 1, 2$.

(ii) For any $\langle S, d \rangle$ and $\langle T, d \rangle$ with $S \subset T$ and $\bar{s}_i = \bar{t}_i$ for i, we have

$$f_j(S, d) \leq f_j(T, d)$$

for $j \neq i$.

Strong monotonicity (STM): For any $\langle S, d \rangle$ and $\langle T, d \rangle$ with $S \subset T$, we have

$$f(S, d) \leq f(T, d).$$

Kalai-Smorodinsky

The unique bargaining solution

$$f^{KS} : B \rightarrow \mathbb{R}^2$$

satisfying SYM, WPO, INV and INM is given by

$$f^{KS}(S,d) = \{\frac{s_1}{\bar{s}_1} = \frac{s_2}{\bar{s}_2} : s \in S\} \cap WPO(S).$$

Proof

Normalize $(d_1, d_2) = (0, 0)$ and define

$$S' = \{(\frac{s_1}{\bar{s}_1}, \frac{s_2}{\bar{s}_2}) : (s_1, s_2) \in S\}$$

and note that $\bar{s}'_i = 1$ for each $i = 1, 2$.

By INV we have that

$$\frac{f_1^{KS}(S)}{\bar{s}_1} = f_1^{KS}(S') = f_2^{KS}(S') = \frac{f_2^{KS}(S)}{\bar{s}_2}.$$

Next, we show that f^{KS} is the only solution that satisfies SYM, WPO, INV and INM.

Let

$$T := co\{(0,0),(1,0),(0,1), f^{KS}(S')\}$$

and note that

$$f^{KS}(T) = f^{KS}(S')$$

and that for any $f(T)$ that satisfies WPO and SYM we have

$$f(T) = f^{KS}(T).$$

By INM we have that

$$f_i(S') \geq f_i(T) = f_i^{KS}(S').$$

for $i = 1, 2$.

By WPO of f^{KS} we know that $f(S') \leq f^{KS}(S')$ and thus

$$f(S') = f^{KS}(S').$$

And, by INV we have that

$$f(S) = f^{KS}(S)$$

which completes the proof.

Kalai

The unique bargaining solution

$$f^{KS} : B \to \mathbb{R}^2$$

satisfying SYM, WPO and STM is given by

$$f^K(S, d) = \max\{(s_1, s_2) \in S : s_1 = s_2\}.$$

Proof

Normalize $(d_1, d_2) = (0, 0)$ and define the symmetric set

$$\begin{aligned} T &= \{s \in S : (s_1, s_2) \in S \\ &\Leftrightarrow (s_2, s_1) \in S, s \leq f^K(S)\}. \end{aligned}$$

For example, the set T can be given by

$$T = \{s \in S : s_1 = s_2\}.$$

For any solution f that satisfies SYM and WPO

$$f(T) = f^K(S).$$

Since $T \subset S$, by STM, $f(T) \leq f(S)$ and thus $f^K(S) \leq f(S)$.

By WPO of $f^K(S) \geq f(S)$ so we have that

$$f^K(S) = f(S)$$

which concludes the proof.

The strategic approach (OR 7)

The players bargain over a pie of size 1.

An agreement is a pair (x_1, x_2) where x_i is player i's share of the pie. The set of possible agreements is

$$X = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$$

Player i prefers $x \in X$ to $y \in X$ if and only if $x_i > y_i$.

The bargaining protocol

The players can take actions only at times in the (infinite) set $T = \{0, 1, 2, ...\}$. In each $t \in T$ player i, proposes an agreement $x \in X$ and $j \neq i$ either accepts (Y) or rejects (N).

If x is accepted (Y) then the bargaining ends and x is implemented. If x is rejected (N) then the play passes to period $t + 1$ in which j proposes an agreement.

At all times players have perfect information. Every path in which all offers are rejected is denoted as disagreement (D). The only asymmetry is that player 1 is the first to make an offer.

Preferences

Time preferences (toward agreements at different points in time) are the driving force of the model.

A bargaining game of alternating offers is

- an extensive game of perfect information with the structure given above, and
- player i's preference ordering $\precsim_i$over $(X \times T) \cup \{D\}$ is complete and transitive.

Preferences over $X \times T$ are represented by $\delta_i^t u_i(x_i)$ for any $0 < \delta_i < 1$ where u_i is increasing and concave.

Assumptions on preferences

A1 <u>Disagreement is the worst outcome</u>

For any $(x, t) \in X \times T$,

$$(x, t) \succsim_i D$$

for each i.

A2 <u>Pie is desirable</u>

– For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t) \text{ if and only if } x_i > y_i.$$

A3 Time is valuable

For any $t \in T$, $s \in T$ and $x \in X$

$$(x,t) \succsim_i (x,s) \text{ if } t < s$$

and with strict preferences if $x_i > 0$.

A4 Preference ordering is continuous

Let $\{(x_n,t)\}_{n=1}^{\infty}$ and $\{(y_n,s)\}_{n=1}^{\infty}$ be members of $X \times T$ for which

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

Then, $(x,t) \succsim_i (y,s)$ whenever $(x_n,t) \succsim_i (y_n,s)$ for all n.

A2-**A4** imply that for any outcome (x, t) either there is a unique $y \in X$ such that

$$(y, 0) \sim_i (x, t)$$

or

$$(y, 0) \succ_i (x, t)$$

for every $y \in X$.

Note $\succsim_i$satisfies **A2**-**A4** iff it can be represented by a continuous function

$$U_i : [0, 1] \times T \longrightarrow \mathbb{R}$$

that is increasing (deceasing) in the first (second) argument.

A5 Stationarity

For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t+1) \text{ if and only if } (x, 0) \succ_i (y, 1).$$

If $\succsim_i$ satisfies **A2-A5** then for every $\delta \in (0, 1)$ there exists a continuous increasing function $u_i : [0, 1] \rightarrow \mathbb{R}$ (not necessarily concave) such that

$$U_i(x_i, t) = \delta_i^t u_i(x_i).$$

Present value

Define $v_i : [0,1] \times T \to [0,1]$ for $i = 1, 2$ as follows

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y, 0) \sim_i (x, t) \\ 0 & \text{if } (y, 0) \succ_i (x, t) \text{ for all } y \in X. \end{cases}$$

We call $v_i(x_i, t)$ player i's present value of (x, t) and note that

$$(y, t) \succ_i (x, s) \text{ whenever } v_i(y_i, t) > v_i(x_i, s).$$

If $\succsim_i$satisfies **A2**-**A4**, then for any $t \in T$ $v_i(\cdot, t)$ is continuous, non decreasing and increasing whenever $v_i(x_i, t) > 0$.

Further, $v_i(x_i, t) \leq x_i$ for every $(x, t) \in X \times T$ and with strict whenever $x_i > 0$ and $t \geq 1$.

With **A5**, we also have that

$$v_i(v_i(x_i, 1), 1) = v_i(x_i, 2)$$

for any $x \in X$.

Delay

A6 Increasing loss to delay

$x_i - v_i(x_i, 1)$ is an increasing function of x_i.

If u_i is differentiable then under **A6** in any representation $\delta_i^t u_i(x_i)$ of $\succsim_i$

$$\delta_i u_i'(x_i) < u_i'(v_i(x_i, 1))$$

whenever $v_i(x_i, 1) > 0$.

This assumption is weaker than concavity of u_i which implies

$$u_i'(x_i) < u_i'(v_i(x_i, 1)).$$

The single crossing property of present values

If $\succsim_i$ for each i satisfies **A2-A6**, then there exist a unique pair $(x^*, y^*) \in X \times X$ such that

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

– For every $x \in X$, let $\psi(x)$ be the agreement for which

$$\psi_1(x) = v_1(x_1, 1)$$

and define $H : X \to \mathbb{R}$ by

$$H(x) = x_2 - v_2(\psi_2(x), 1).$$

– The pair of agreements x and $y = \psi(x)$ satisfies also $x_2 = v_2(\psi_2(x), 1)$ iff $H(x) = 0$.

– Note that $H(0,1) \geq 0$ and $H(1,0) \leq 0$, H is a continuous function, and

$$\begin{aligned} H(x) &= [v_1(x_1, 1) - x_1] + \\ &\quad +[1 - v_1(x_1, 1) - v_2(1 - v_1(x_1, 1), 1)]. \end{aligned}$$

– Since $v_1(x_1, 1)$ is non decreasing in x_1, and both terms are decreasing in x_1, H has a unique zero by **A6**.

Examples

[1] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = \delta_i^t x_i$$

where $\delta_i \in (0, 1)$, and $U_i(D) = 0$.

[2] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = x_i - c_i t$$

where $c_i > 0$, and $U_i(D) = -\infty$ (constant cost of delay).

Although **A6** is violated, when $c_1 \neq c_2$ there is a unique pair $(x, y) \in X \times X$ such that $y_1 = v_1(x_1, 1)$ and $x_2 = v_2(y_2, 1)$.

Strategies

Let X^t be the set of all sequences $\{x^0, ..., x^{t-1}\}$ of members of X.

A strategy of player 1 (2) is a sequence of functions

$$\sigma = \{\sigma^t\}_{t=0}^{\infty}$$

such that $\sigma^t : X^t \to X$ if t is even (odd), and $\sigma^t : X^{t+1} \to \{Y, N\}$ if t is odd (even).

The way of representing a player's strategy in closely related to the notion of automation.

Nash equilibrium

For any $\bar{x} \in X$, the outcome $(\bar{x}, 0)$ is a NE when players' preference satisfy **A1**-**A6**.

To see this, consider the stationary strategy profile

Player 1	proposes	$\bar{x}$
	accepts	$x_1 \geq \bar{x}_1$
Player 2	proposes	$\bar{x}$
	accepts	$x_2 \geq \bar{x}_2$

This is an example for a pair of one-state automate.

The set of outcomes generated in the Nash equilibrium includes also delays (agreements in period 1 or later).

Subgame perfect equilibrium

Any bargaining game of alternating offers in which players' preferences satisfy **A1**-**A6** has a unique SPE which is the solution of the following equations

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

Note that if $y_1^* > 0$ and $x_2^* > 0$ then

$$(y_1^*, 0) \sim_1 (x_1^*, 1) \text{ and } (x_2^*, 0) \sim_2 (y_2^*, 1).$$

The equilibrium strategy profile is given by

Player 1	proposes	x^*
	accepts	$y_1 \geq y_1^*$
Player 2	proposes	y^*
	accepts	$x_1 \leq x_1^*$

The unique outcome is that player 1 proposes x^* in period 0 and player 2 accepts.

Step 1 (x^*, y^*) is a SPE

Player 1:

- proposing x^* at t^* leads to an outcome (x^*, t^*). Any other strategy generates either

$$(x, t) \text{ where } x_1 \leq x_1^* \text{ and } t \geq t^*$$

or

$$(y^*, t) \text{ where } t \geq t^* + 1$$

or D.

- Since $x_1^* > y_1^*$ it follows from **A1**-**A3** that (x^*, t^*) is a best response.

Player 2:

– accepting x^* at t^* leads to an outcome (x^*, t^*). Any other strategy generates either

$$(y, t) \text{ where } y_2 \leq y_2^* \text{ and } t \geq t^* + 1$$

or

$$(x^*, t) \text{ where } t \geq t^*$$

or D.

– By **A1**-**A3** and **A5**

$$(x^*, t^*) \succsim_2 (y^*, t^* + 1)$$

and thus accepting x^* at t^*, which leads to the outcome (x^*, t^*), is a best response.

Note that similar arguments apply to a subgame starting with an offer of player 2.

Step 2 (x^*, y^*) is the unique SPE

Let G_i be a subgame starting with an offer of player i and define

$$M_i = \sup\{v_i(x_i, t) : (x, t) \in SPE(G_i)\},$$

and

$$m_i = \inf\{v_i(x_i, t) : (x, t) \in SPE(G_i)\}.$$

It is suffices to show that

$$M_1 = m_1 = x_1^* \text{ and } M_2 = m_2 = y_2^*.$$

It follows that the present value for player 1 (2) of every SPE of G_1 (G_2) is x_1^* (y_2^*).

First, we argue that in every SPE of G_1 and G_2 the first offer is accepted because

$$v_1(y_1^*, 1) \leq y_1^* < x_1^* \text{ and } v_2(x_2^*, 1) \leq x_2^* < y_2^*$$

(after a rejection, the present value for player 1 is less than x_1^* and for player 2 is less than y_2^*).

It remains to show that

$$m_2 \geq 1 - v_1(M_1, 1) \tag{3}$$

and

$$M_1 \leq 1 - v_2(m_2, 1). \tag{4}$$

[3] and the fact that $m_2 \leq y_2^*$ imply that the pair $(M_1, 1-m_2)$ lies below the line

$$y_1 = v_1(x_1, 1),$$

and [4] and the fact that $M_1 \leq x_1^*$ imply that this pair lies to the left of the line

$$x_2 = v_2(y_2, 1).$$

Thus,

$$M_1 = x_1^* \text{ and } m_2 = y_2^*,$$

and with the role of the players reversed, the same argument shows that $M_2 = y_2^*$ and $m_1 = x_1^*$.

Properties of Rubinstein's model

[1] Delay (without uncertainty)

Subgame perfection alone cannot not rule out delay. In Rubinstein's model delay is closely related to the existence of multiple equilibria.

The uniqueness proof relies only on **A1**-**A3** and **A6**. When both players have the same constant cost of delay (**A6** is violated), there are multiple equilibria.

If the cost of delay is small enough, in some of these equilibria, agreement is not reached immediately. Any other conditions that guarantees a unique solution can be used instead of **A6**.

An example

Assume that $X = \{a, b, c\}$ where $a_1 > b_1 > c_1$, the ordering $\succsim_i$ satisfies **A1**-**A3** and **A5** for $i = 1, 2$, and if $(x, t) \succ (y, t)$ then $(x, t+1) \succ (y, t)$.

Then, for each $\bar{x} \in X$, the pair of strategies in which each player insists on $\bar{x}$

Player 1	proposes	$\bar{x}$
	accepts	$x_1 \geq \bar{x}_1$
Player 2	proposes	$\bar{x}$
	accepts	$x_2 \geq \bar{x}_2$

is a subgame perfect equilibrium.

An example of a subgame perfect equilibrium in which agreement is reached in period 1 is given by

		A	B	C
Player 1	proposes	a	b	c
	accepts		a and b	a, b, and c
Player 2	proposes		b	c
	accepts	c	b and c	c

where A is the initial state, B and C are absorbing states, and if player 2 rejects a (b or c) then the state changes to B (C).

The outcome is that player 1 offers a in period 0, player 2 rejects and proposes b in period 1 which player 1 accepts.

[2] Patience

The ordering $\succsim_1'$ is *less patient than* $\succsim_1$ if

$$v_1'(x_1, 1) \leq v_1(x_1, 1)$$

for all $x \in X$ (with constant cost of delay $\delta_1' \leq \delta_1$).

The models predicts that when a player becomes less patient his negotiate share of the pie decreases.

[3] Asymmetry

The structure of the model is asymmetric only in one respect: player 1 is the first to make an offer.

Recall that with constant discount rates the equilibrium condition implies that

$$y_1^* = \delta_1 x_1^* \text{ and } x_2^* = \delta_2 y_2^*$$

so that

$$x^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right) \text{ and } y^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right).$$

Thus, if $\delta_1 = \delta_2 = \delta$ $(v_1 = v_2)$ then

$$x^* = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) \text{ and } y^* = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$$

so player 1 obtains more than half of the pie.

By shrinking the length of a period by considering a sequence of games indexed by Δ in which $u_i = \delta_i^{\Delta t} x_i$ we have

$$\lim_{\Delta \to 0} x^*(\Delta) = \lim_{\Delta \to 0} y^*(\Delta) = \left(\frac{\log \delta_2}{\log \delta_1 + \log \delta_2}, \frac{\log \delta_1}{\log \delta_1 + \log \delta_2}\right)$$

(l'Hôpital's rule).