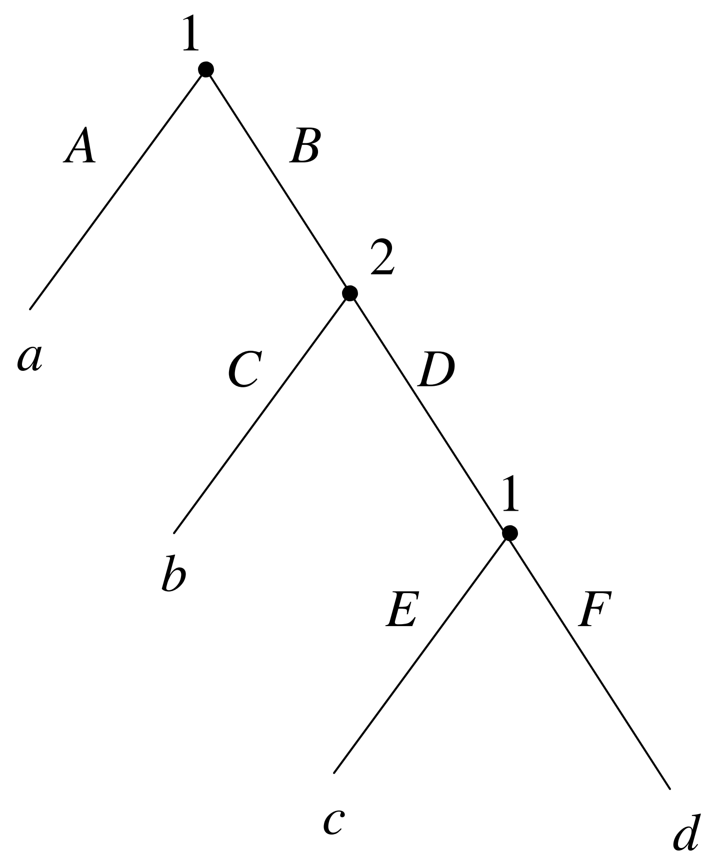
Economics 201B
Economic Theory
(Spring 2022)
Extensive Games with Perfect and Imperfect Information

Topics: perfect information (OR 6.1), subgame perfection (OR 6.2), forward induction (OR 6.6), imperfect information (OR 11.1), mixed and behavioral strategies (OR 11.4), sequential equilibrium (OR 12.2), trembling hand perfection (OR 12.4).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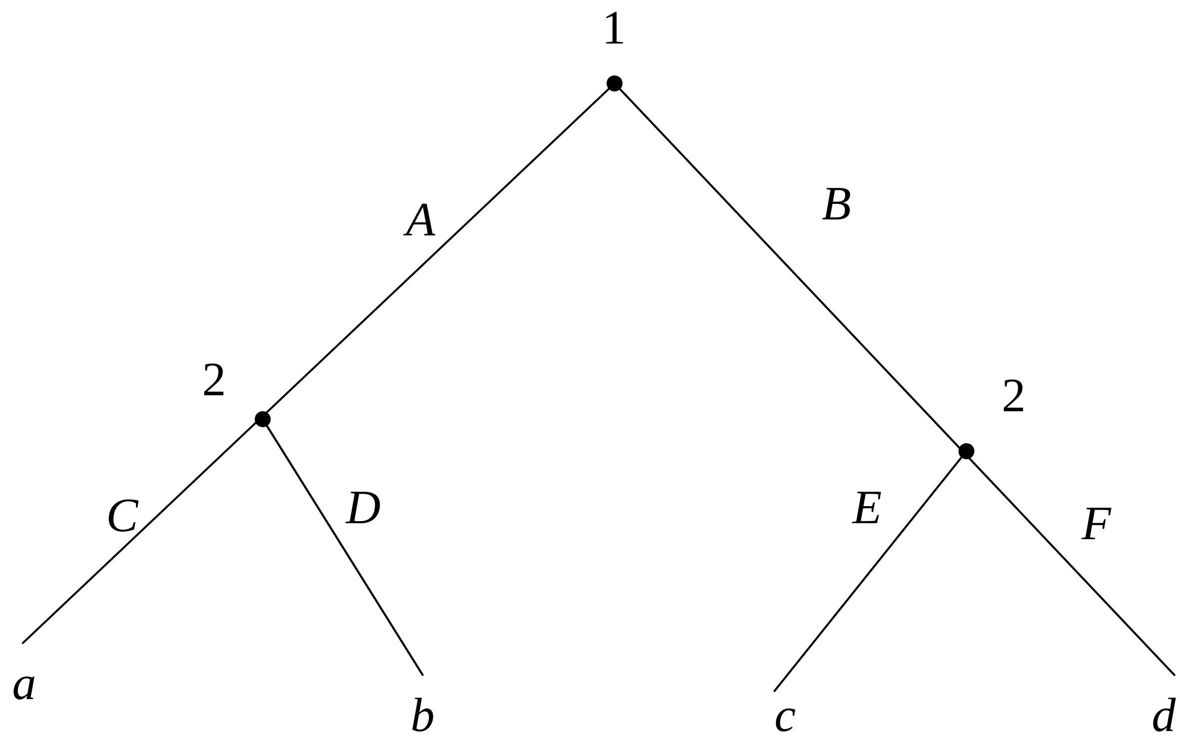

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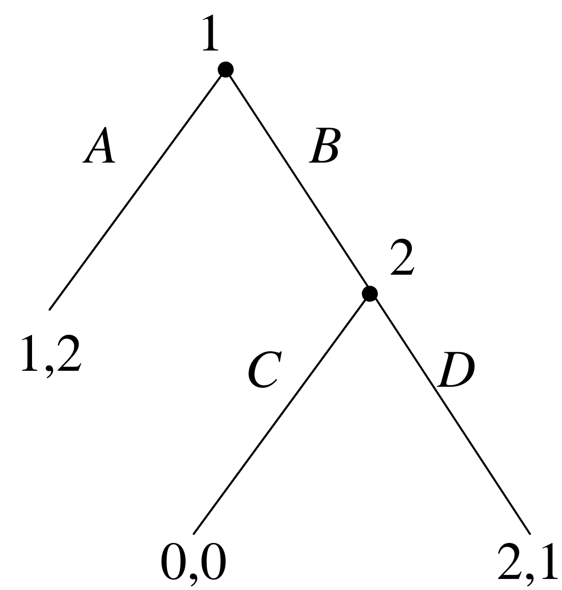
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Perfect information (OR 6.1)

A finite extensive game with perfect information $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ consists of

- A set N of players.
- A set H of sequences (histories) where $\varnothing \in H$ and for any $L < K$

$$(a^k)_{k=1}^{K} \in H \Longrightarrow (a^k)_{k=1}^{L} \in H.$$

- A player function $P : H \backslash Z \to N$ where $h \in Z \subseteq H$ if $(h, a) \notin H$.
- A preference relation $\succsim_i$ on Z for each player $i \in N$.

Strategies, outcomes and Nash equilibrium

A strategy

$$s_i : h \longrightarrow A(h) \text{ for every } h \in H \backslash Z \text{ such that } P(h) = i.$$

A Nash equilibrium of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ is a strategy profile $(s_i^*)_{i \in N}$ such that for any $i \in N$

$$O(s^*) \succsim_i O(s_i, s_{-i}^*) \ \forall s_i$$

where $O(s) = (a^1, ..., a^K) \in Z$ such that

$$s_{P(a^1,...,a^k)}(a^1, ..., a^k) = a^{k+1}$$

for any $0 \leq k < K$ (an outcome).

The (reduced) strategic form

$G = \left\langle N, (S_i), (\succsim_i') \right\rangle$ is the strategic form of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ if for each $i \in N$, S_i is player i's strategy set in Γ and $\succsim_i'$ is defined by

$$s \succsim_i' s' \Leftrightarrow O(s) \succsim_i O(s') \; \forall s, s' \in \times_{i \in N} S_i$$

$G = \left\langle N, (S_i'), (\succsim_i'') \right\rangle$ is the reduced strategic form of $\Gamma = \langle N, H, P, (\succsim_i) \rangle$ if for each $i \in N$, S_i' contains one member of *equivalent* strategies in S_i, that is,

$$s_i, s_i' \in S_i \text{ are equivalent if } (s_i, s_{-i}) \sim_j' (s_i', s_{-i}) \forall j \in N,$$

and $\succsim_i''$ defined over $\times_{j \in N} S_j'$ and induced by $\succsim_i'$.

Subgames and subgame perfection (OR 6.2)

A subgame of Γ that follows the history h is the game $\Gamma(h)$

$$\langle N, H|_h, P|_h, (\succsim_i|_h)\rangle$$

where for each $h' \in H_h$

$(h, h') \in H, P|_h(h') = P(h, h')$ and $h' \succsim_i|_h h'' \Leftrightarrow (h, h') \succsim_i (h, h'')$.

$s^* \in \times_{i \in N} S_i$ is a subgame perfect equilibrium (SPE) of Γ if

$$O_h(s_i^*|_h, s_{-i}^*|_h) \succsim_i|_h O_h(s_i|_h, s_{-i}^*|_h)$$

for each $i \in N$ and $h \in H \backslash Z$ for which $P(h) = i$ and for any $s_i|_h$.

Thus, the equilibrium of the full game must induce on equilibrium on every subgame.

Backward induction and Kuhn's theorems

Let Γ be a finite extensive game with perfect information

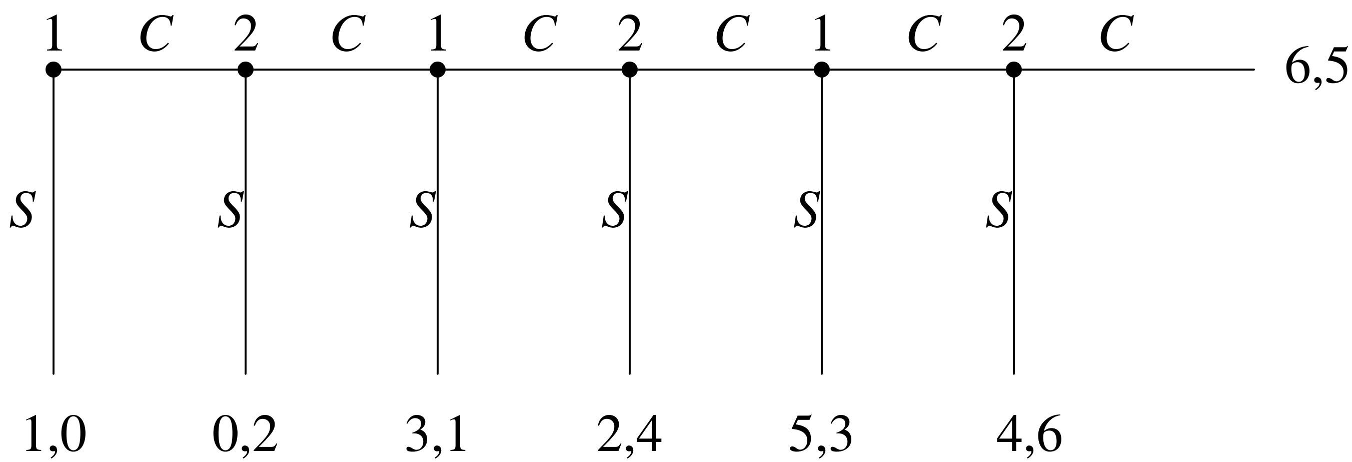
– Γ has a SPE (Kuhn's theorem).

  The proof is by backward induction (Zermelo, 1912) which is also an algorithm for calculating the set of SPE.

– Γ has a unique SPE if there is no $i \in N$ such that $z \sim_i z'$ for any $z, z' \in Z$.

– Γ is dominance solvable if $z \sim_i z' \;\; \exists i \in N$ then $z \sim_j z' \; \forall j \in N$ (but elimination of weakly dominated strategies in G may eliminate the SPE in Γ).

OR 107.1 (the centipede game)

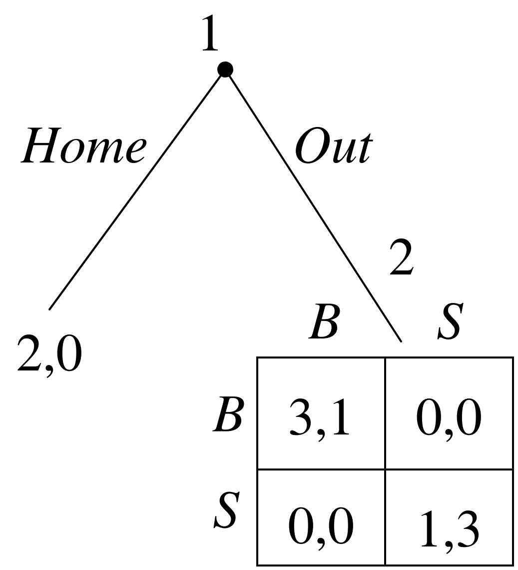


Forward induction (OR 6.6)

- Backward induction cannot always ensure a self-enforcing equilibrium (forward and backward induction).

- In an extensive game with simultaneous moves, players interpret a deviation as a signal about future play.

- The concept of iterated weak dominance can be used to capture forward and backward induction.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Imperfect information (OR 11.1)

An extensive game with imperfect information

$$\Gamma = \langle N, H, P, f_c, (\mathcal{I}_i)_{i \in N}, (\succsim_i) \rangle$$

consists of

– a probability measure $f_c(\cdot\,|h)$ on $A(h)$ for all h such that $P(h) = c$ (chance determines the action taken after the history h), and

– an information partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$ for every $i \in N$ such that

$$A(h) = A(h')$$

whenever $h, h' \in I_i$ (an information set).

Perfect and imperfect recall

Let $X_i(h)$ be player i's experience along the history h:

- all I_i encountered,
- actions $a_i \in A(I_i)$ taken at them, and
- the order that these events occur.

An extensive game with imperfect information has perfect recall if for each $i \in N$

$$X_i(h) = X_i(h')$$

whenever $h, h' \in I_i$.

Pure, mixed and behavioral strategies (OR 11.4)

In an extensive game $\langle N, H, P, f_c, (\mathcal{I}_i)_{i \in N}, (\succsim_i) \rangle$, for player $i \in N$

- a <u>pure</u> strategy assigns and action $a_i \in A(I_i)$ to each information set $I_i \in \mathcal{I}_i$,
- a <u>mixed</u> strategy is a probability measure over the set of pure strategies, and
- a <u>behavioral</u> strategy is a collection of independent probability measures $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$.

For any $\sigma = (\sigma_i)_{i \in N}$ (mixed or behavioral) an outcome $O(\sigma)$ is a probability distribution over z that results from σ.

Outcome-equivalent strategies

Two strategies (mixed or behavioral) of player i, σ_i and σ'_i, are outcome equivalent if

$$O(\sigma_i, s_{-i}) = O(\sigma'_i, s_{-i})$$

for every collection s_{-i} of pure strategies.

In any finite game with perfect recall, any mixed strategy of a player has an outcome-equivalent behavioral strategy (the converse is true for a set of games that includes all those with perfect recall).

Strategies and beliefs (OR 12.1)

- Under imperfect information, an equilibrium should specify actions and beliefs about the history that occurred (an assessment).

- An assessment thus consists of a profile of behavioral strategies and a belief system (a probability measure for each information set).

- An assessment is sequentially rational if for each information set, the strategy is a best response given the beliefs.

Consistency of the players' beliefs:

(i) derived from strategies using Bayes' rule

(ii) derived from some alternative strategy profile using Bayes' rule at information sets that need not be reached

(iii) all players share the same beliefs about the cause of any unexpected event.

Sequential equilibrium (OR 12.2)

An assessment (β, μ) is sequentially rational if for each $i \in N$ and every $I_i \in \mathcal{I}_i$

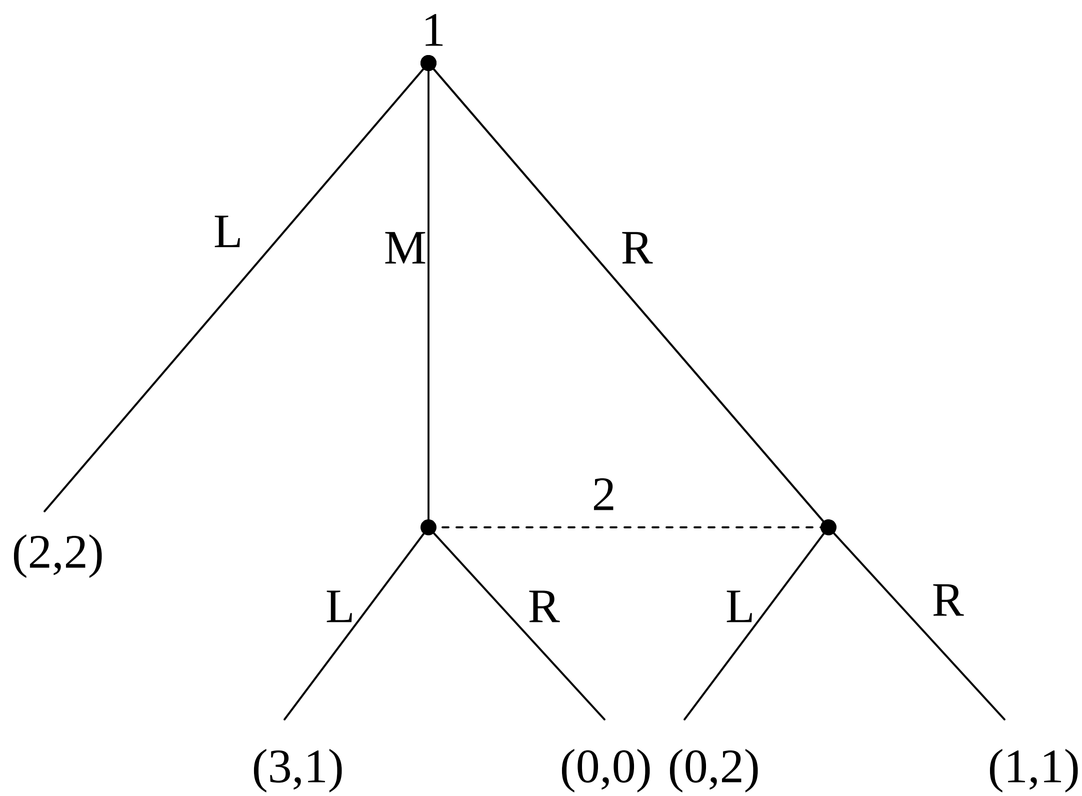
$$O(\beta, \mu \,|\, I_i) \succsim_i O((\beta_i', \beta_{-i}), \mu \,|\, I_i) \text{ for all } \beta_i'.$$

(β, μ) is consistent if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty} \to (\beta, \mu)$ such that for each n:

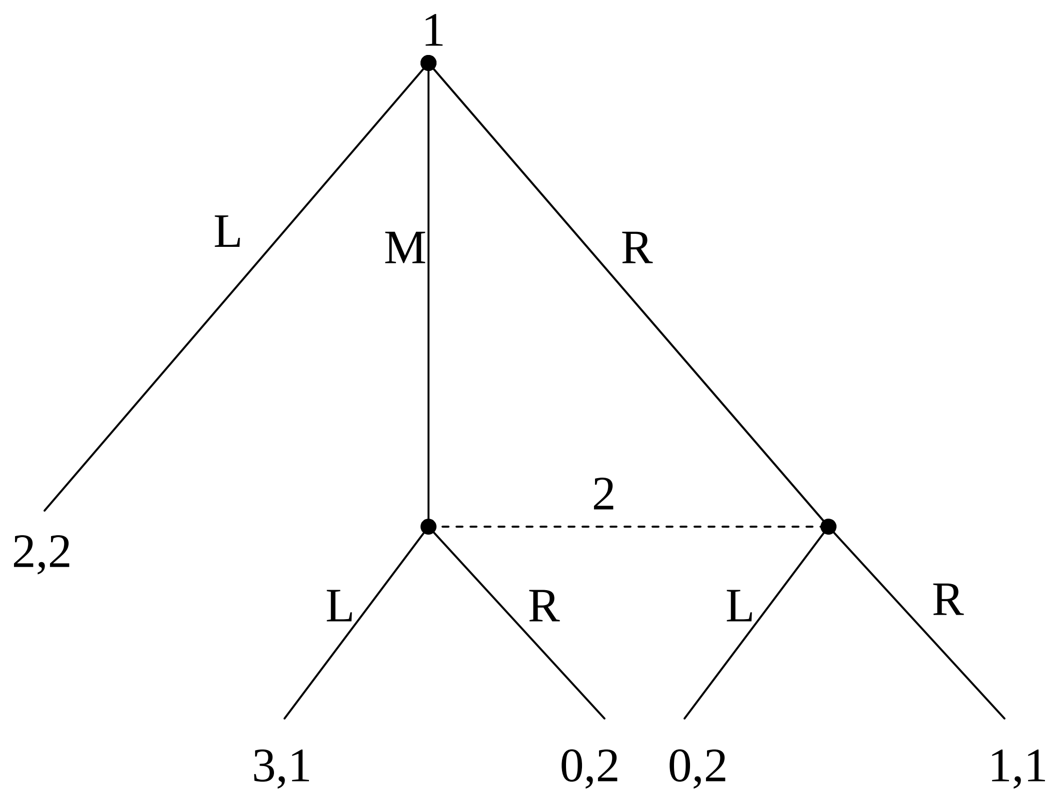
– β^n is completely (strictly) mixed and μ^n is derived from β^n using Bayes' rule.

(β, μ) is a sequential equilibrium if it is sequentially rational and consistent (Kreps and Wilson, 1982).

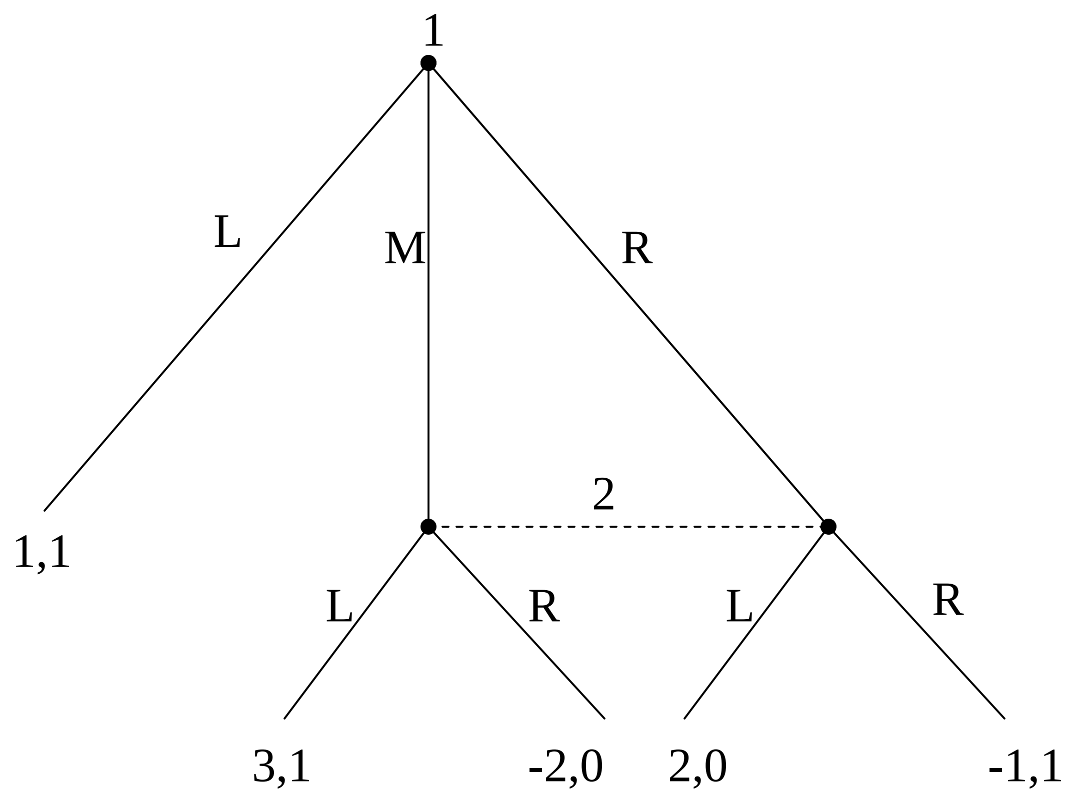
<u>OR 219.1</u>



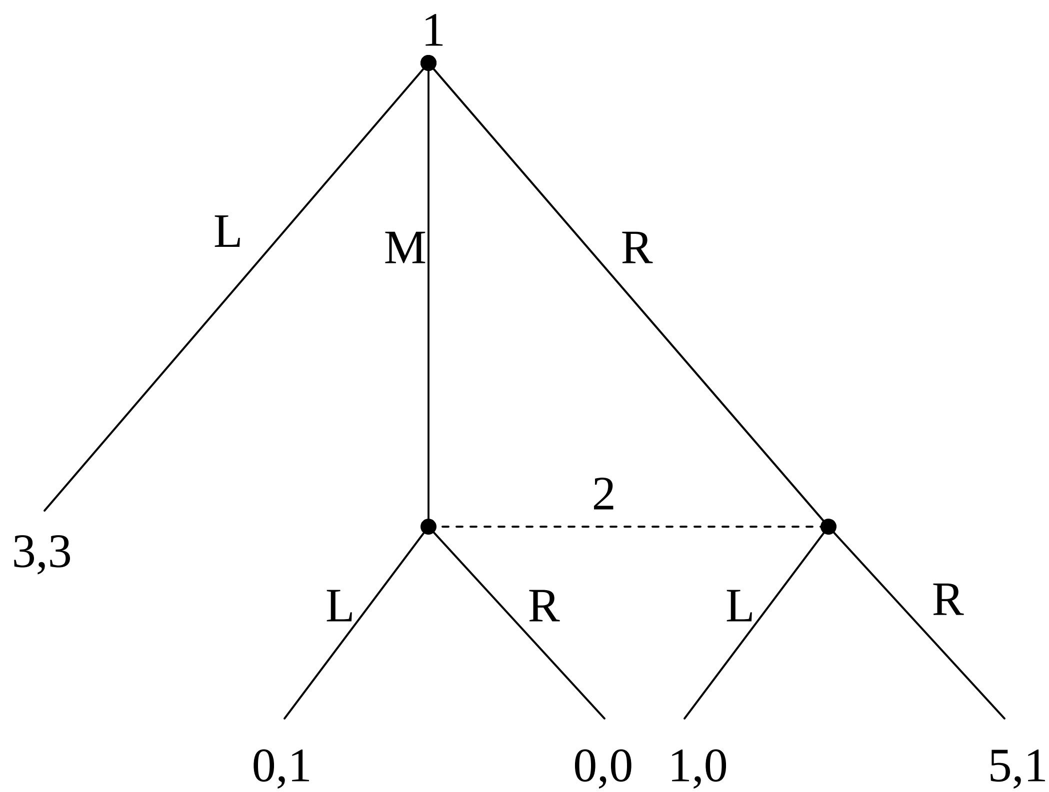
OR 220.1



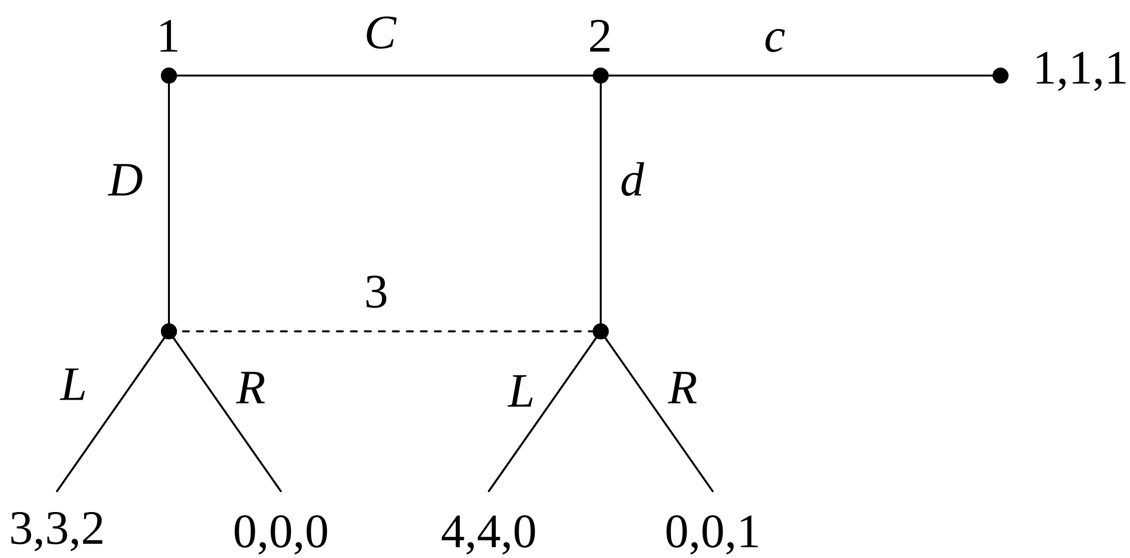
OR 226.1



OR 227.1



OR 225.1 (Selten's horse)



Perfect Bayesian equilibrium (OR 12.3)

A Bayesian extensive game $\langle \Gamma, (\Theta_i), (p_i), (u_i) \rangle$ is a game with observable actions where

- Γ is an extensive game of perfect information and simultaneous moves,

- Θ_i is a finite set of possible types of player i,

- p_i is a probability distribution on Θ_i for which $p_i(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, and

- $u_i : \Theta_i \times Z \to \mathbb{R}$ is a vNM utility function.

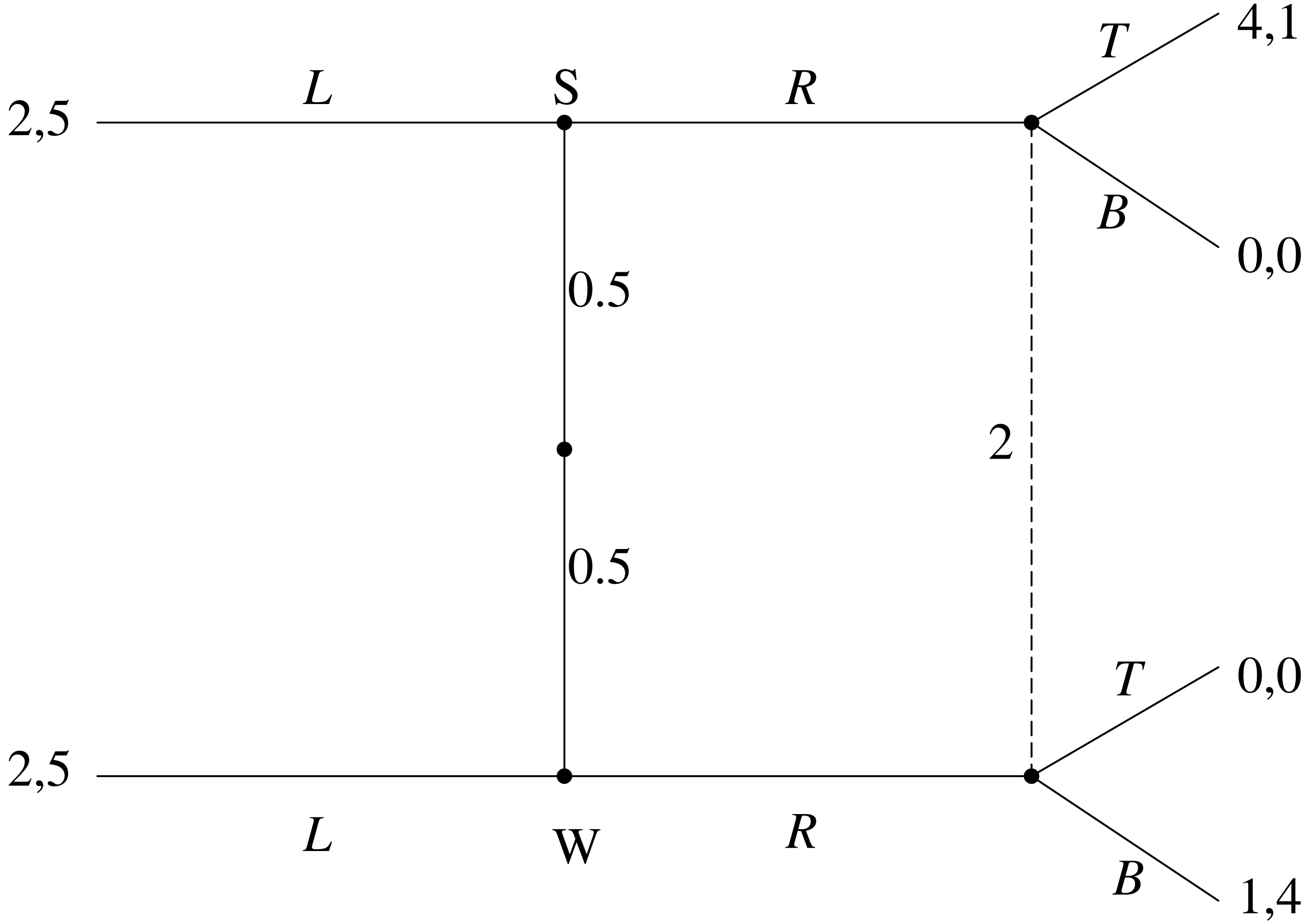
Let $\sigma_i(\theta_i)$ be a behavioral statuary of player i of type θ_i and $\mu_{-i}(h)$ be a probability measure over Θ_i.

(σ, μ) is sequentially rational if for every $h \in H \backslash Z$, $i \in P(h)$ and $\theta_i \in \Theta_i$

$$O(\sigma, \mu_{-i} \,|h) \succsim_i O((\sigma_i', \sigma_{-i}), \mu_{-i} \,|h) \ \forall \sigma_i'$$

(σ, μ) is PB-consistent if for each $i \in N$ $\mu_{-i}(\emptyset) = p_i$ (correct initial beliefs) and μ_{-i} is derived from p_i and $a_i \in A(h)$ via Bayes' rule (action-determined beliefs) when possible.

(σ, μ) is a perfect Bayesian equilibrium (PBE) if it is sequentially rational and PB-consistent.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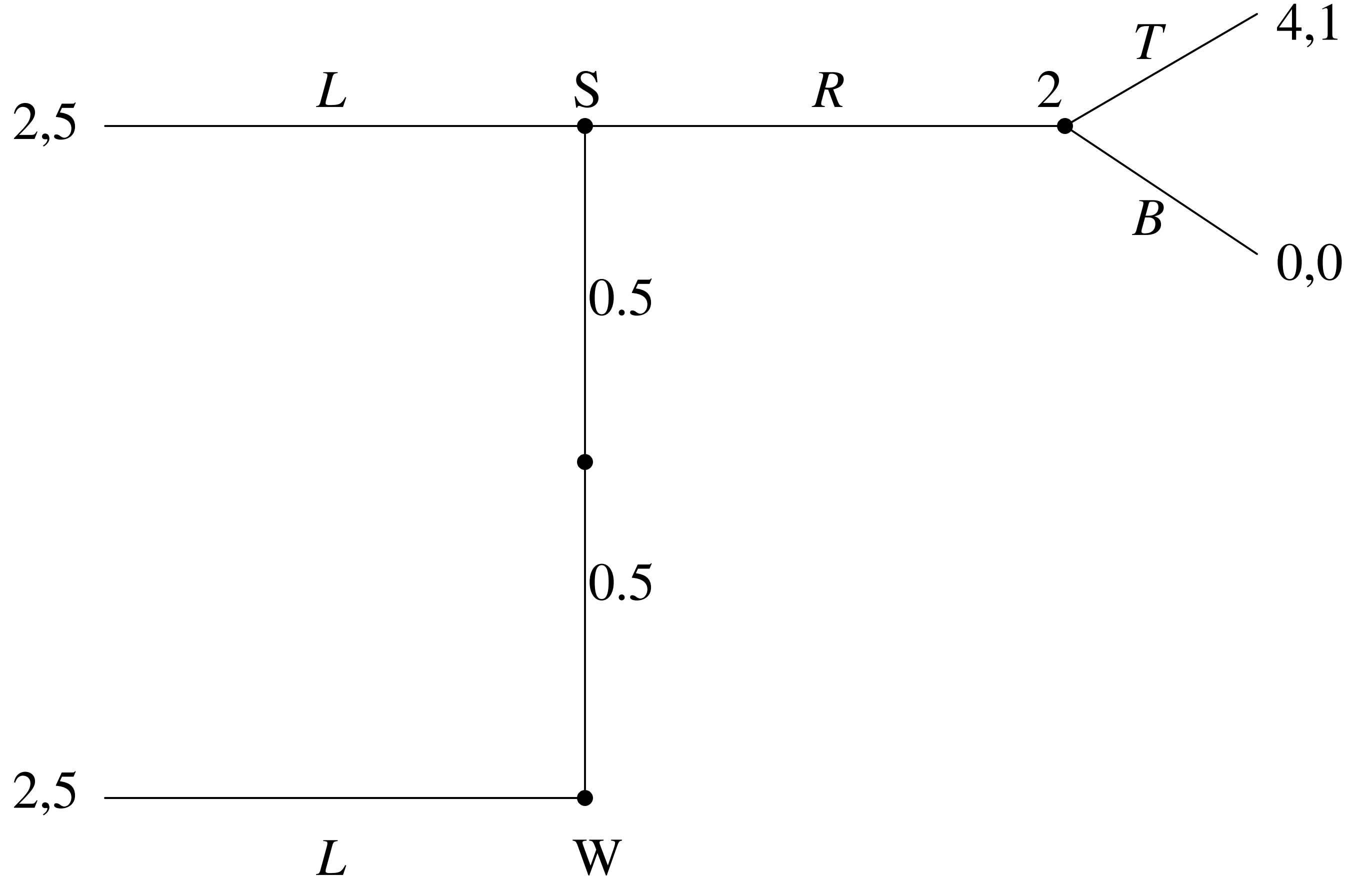
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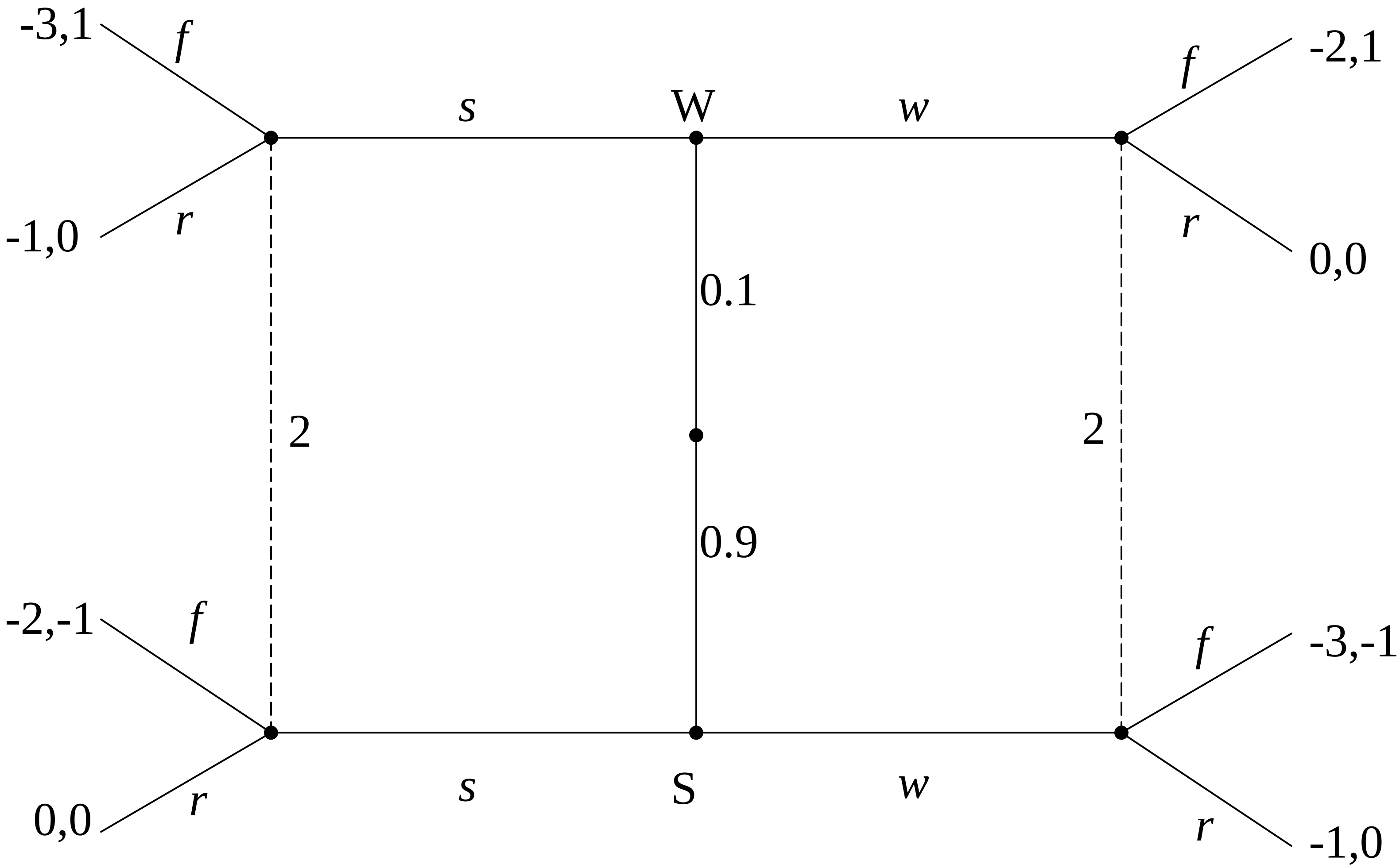

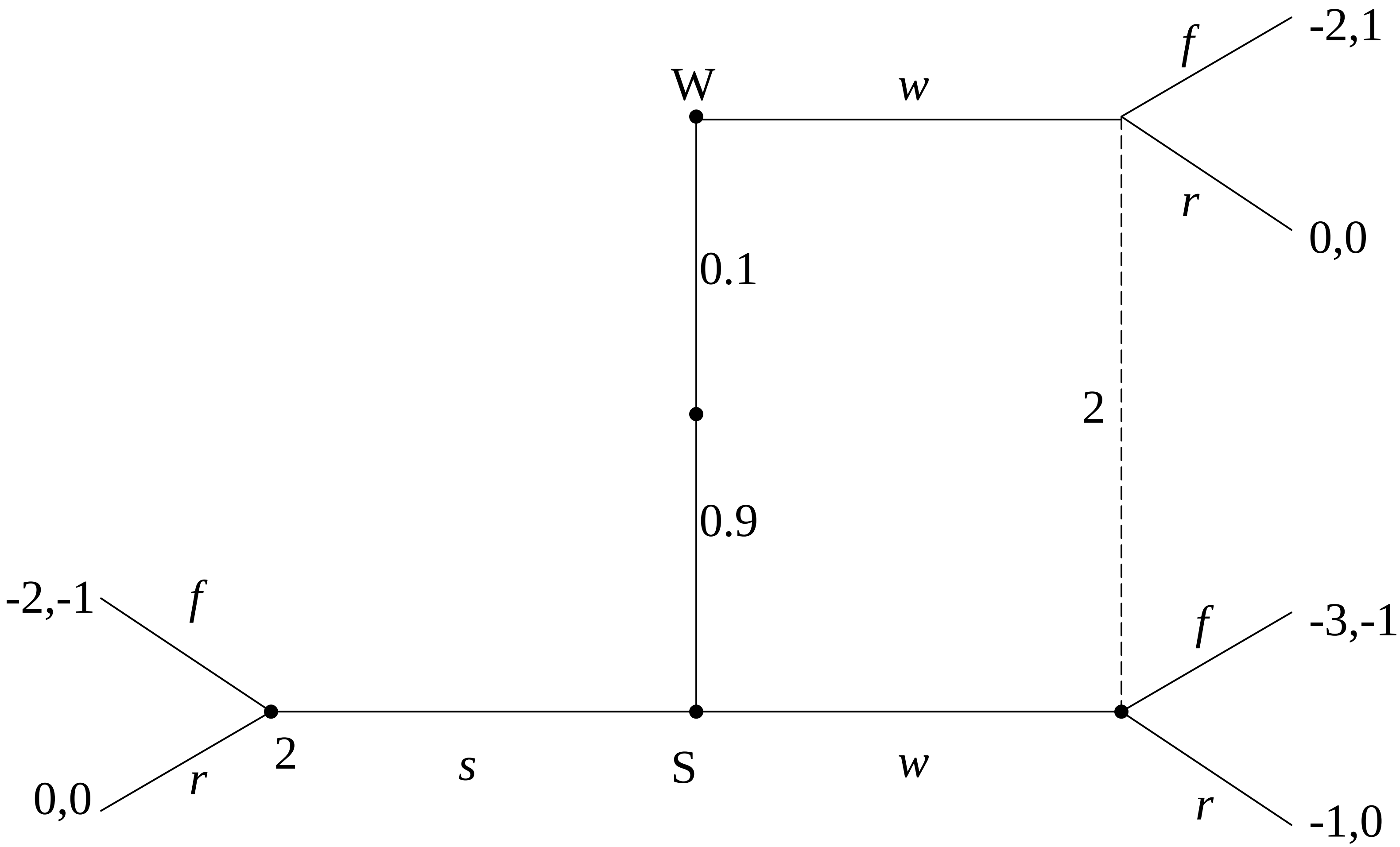
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r
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Trembling hand perfection (OR 12.4)

Trembling hand perfect equilibrium (THP) excludes strategies that are "unsafe" given the risk of small mistakes.

σ is a THP of a finite strategic game if $\exists(\sigma^k)_{k=1}^{\infty}$ of completely mixed strategy profiles such that

- $(\sigma^k)_{k=1}^{\infty}$ converges to σ, and
- $\sigma_i \in BR_i(\sigma_{-i}^k)$ for each i and all k.

A THP of a finite extensive game is a behavioral strategy profile β that corresponds to a THP of the agent strategic form of the game.