Economics 209A
Theory and Application of Non-Cooperative Games
(Fall 2013)

Bargaining
The axiomatic approach
OR 15

Bargaining problem

Nash's (1950) work is the starting point for formal bargaining theory.

The bargaining problem consists of

- a set of utility pairs that can be derived from possible agreements, and
- a pair of utilities which is designated to be a disagreement point.

Bargaining solution

The bargaining solution is a function that assigns a unique outcome to every bargaining problem.

Nash's bargaining solution is the first solution that

- satisfies four plausible conditions, and
- has a simple functional form, which make it convenient to apply.

A bargaining situation

A bargaining situation is a tuple $\langle N, A, D, (\succsim_i) \rangle$ where

- N is a set of players or bargainers ($N = \{1, 2\}$),
- A is a set of agreements/outcomes,
- D is a disagreement outcome, and
- $\succsim_i$ is a preference ordering over the set of lotteries over $A \cup \{D\}$.

The objects N, A, D and $\succsim_i$ for $i = \{1, 2\}$ define a bargaining situation.

$\succsim_1$ and $\succsim_2$ satisfy the assumption of vNM so for each i there is a utility function $u_i : A \cup \{D\} \to \mathbb{R}$.

$\langle S, d \rangle$ is the primitive of Nash's bargaining problem where

- $S = (u_1(a), u_2(a))$ for $a \in A$ the set of all utility pairs, and
- $d = (u_1(D), u_2(D))$.

A <u>bargaining problem</u> is a pair $\langle S, d\rangle$ where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$ and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems $\langle S, d\rangle$ is denoted by B.

A <u>bargaining solution</u> is a function $f : B \rightarrow \mathbb{R}^2$ such that f assigns to each bargaining problem $\langle S, d\rangle \in B$ a unique element in S.

The definitions of the bargaining problem and solution have few restrictions (the convexity assumption on S is more technical):

- bargaining situations that induce the same pair $\langle S, d \rangle$ are treated identically,

- the utilities obtainable in the outcome of bargaining are limited since S is bounded,

- players can agree to disagree since $d \in S$ and there is an agreement preferred by both players to the disagreement outcome.

Nash's axioms

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely - Nash 1953 -

Does not capture the details of a specific bargaining problem (e.g. alternating or simultaneous offers).

Rather, the approach consists of the following four axioms: invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and (weak) Pareto efficiency.

Invariance to equivalent utility representations (INV)

$\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ if

$$d_i' = \alpha_i d_i + \beta_i$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Note that if $\alpha_i > 0$ for $i = 1, 2$ then $\langle S', d' \rangle$ is itself a bargaining problem.

If $\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ where $\alpha_i > 0$ for each i, then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$$

for $i = 1, 2$. Hence, $\langle S', d' \rangle$ and $\langle S, d \rangle$ represent the same situation.

INV requires that the utility outcome of the bargaining problem co-vary with representation of preferences.

The physical outcome predicted by the bargaining solution is the same for $\langle S', d' \rangle$ and $\langle S, d \rangle$.

A corollary of INV is that we can restrict attention to $\langle S, d \rangle$ such that

$S \subset \mathbb{R}^2_+$,

$S \cap \mathbb{R}^2_{++} \neq \emptyset$, and

$d = (0, 0) \in S$ (reservation utilities).

Symmetry (SYM)

A bargaining problem $\langle S, d \rangle$ is symmetric if $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$. If the bargaining problem $\langle S, d \rangle$ is symmetric then

$$f_1(S, d) = f_2(S, d)$$

Nash does not describe differences between the players. All asymmetries (in the bargaining abilities) must be captured by $\langle S, d \rangle$.

Hence, if players are the same the bargaining solution must assign the same utility to each player.

Independence of irrelevant alternatives (IIA)

If $\langle S, d \rangle$ and $\langle T, d \rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$ then

$$f(S, d) = f(T, d)$$

If T is available and players agree on $s \in S \subset T$ then they agree on the same s if only S is available.

IIA excludes situations in which the fact that a certain agreement is available influences the outcome.

Weak Pareto efficiency (WPO)

If $\langle S, d \rangle$ is a bargaining problem where $s \in S$ and $t \in S$, and $t_i > s_i$ for $i = 1, 2$ then $f(S, d) \neq s$.

In words, players never agree on an outcome s when there is an outcome t in which both are better off.

Hence, players never disagree since by assumption there is an outcome s such that $s_i > d_i$ for each i.

$\underline{SYM \text{ and } WPO}$

restrict the solution on single bargaining problems.

$\underline{INV \text{ and } IIA}$

requires the solution to exhibit some consistency across bargaining problems.

Nash 1953: there is precisely one bargaining solution, denoted by $f^N(S, d)$, satisfying SYM, WPO, INV and IIA.

Nash's solution

The unique bargaining solution $f^N : B \to \mathbb{R}^2$ satisfying SYM, WPO, INV and IIA is given by

$$f^N(S,d) = \underset{(d_1,d_2)\le(s_1,s_2)\in S}{\arg\max} (s_1 - d_1)(s_2 - d_2)$$

and since we normalize $(d_1, d_2) = (0, 0)$

$$f^N(S,0) = \underset{(s_1,s_2)\in S}{\arg\max} s_1 s_2$$

The solution is the utility pair that maximizes the product of the players' utilities.

Proof

Pick a compact and convex set $S \subset \mathbb{R}^2_+$ where $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

Step 1: f^N is well defined.

- Existence: the set S is compact and the function $f = s_1 s_2$ is continuous.
- Uniqueness: f is strictly quasi-conacave on S and the set S is convex.

Step 2: f^N is the only solution that satisfies SYM, WPO, INV and IIA.

Suppose there is another solution f that satisfies SYM, WPO, INV and IIA.

Let

$$S' = \{(\frac{s_1}{f_1^N(S)}, \frac{s_2}{f_2^N(S)}) : (s_1, s_2) \in S\}$$

and note that $s'_1 s'_2 \leq 1$ for any $s' \in S'$, and thus $f^N(S', 0) = (1, 1)$.

Since S' is bounded we can construct a set T that is symmetric about the 45° line and contains S'

$$T = \{(a, b) : a + b \leq 2\}$$

By WPO and SYM we have $f(T, 0) = (1, 1)$, and by IIA we have $f(S', 0) = f(T, 0) = (1, 1)$.

By INV we have that $f(S', 0) = f^N(S', 0)$ if and only if $f(S, 0) = f^N(S, 0)$ which completes the proof.

Is any axiom superfluous?

$\underline{INV}$

The bargaining solution given by the maximizer of

$$g(s_1, s_2) = \sqrt{s_1} + \sqrt{s_2}$$

over $\langle S, 0 \rangle$ where $S := co\{(0,0),(1,0),(0,2)\}$.

This solution satisfies WPO, SYM and IIA (maximizer of an increasing function). The maximizer of g for this problem is $(1/3, 4/3)$ while $f^N = (1/2, 1)$.

$\underline{SYM}$

The family of solutions $\{f^\alpha\}_{\alpha\in(0,1)}$ over $\langle S, 0\rangle$ where

$$f^\alpha(S,d) = \underset{(d_1,d_2)\leq(s_1,s_2)\in S}{\arg\max} (s_1 - d_1)^\alpha (s_2 - d_2)^{1-\alpha}$$

is called the asymmetric Nash solution.

Any f^α satisfies INV, IIA and WPO by the same arguments used for f^N.

For $\langle S, 0\rangle$ where $S := co\{(0,0),(1,0),(0,1)\}$ we have $f^\alpha(S,0) = (\alpha, 1-\alpha)$ which is different from f^N for any $\alpha \neq 1/2$.

$\underline{WPO}$

Consider the solution f^d given by $f^d(S,d) = d$ which is different from f^N. f^d satisfies INV, SYM and IIA.

WPO in the Nash solution can be replaced with strict individual rationality (SIR) $f(S,d) >> d$.

An application - risk aversion

Dividing a dollar: the role of risk aversion: Suppose that

$$A = \{(a_1, a_2) \in \mathbb{R}^2_+ : a_1 + a_2 \leq 1\}$$

(all possible divisions), $D = (0, 0)$ and for all $a, b \in A$ $a \succsim_i b$ if and only if $a_i \geq b_i$.

Player i's preferences over $A \cup D$ can be represented by $u_i : [0, 1] \to \mathbb{R}$ where each u_i is concave and (WLOG) $u_i(0) = 0$.

Then,

$$S = \{(s_1, s_2) \in \mathbb{R}^2_+ : (s_1, s_2) = (u_1(a_1), u_2(a_2))\}$$

for some $(a_1, a_2) \in A$ is compact and convex and

$$d = (u_1(0), u_2(0)) = (0, 0) \in S.$$

First, note that when $u_1(a) = u_2(a)$ for all $a \in (0, 1]$ then $\langle S, d \rangle$ is symmetric so by SYM and WPO the Nash solution is $(u(1/2), u(1/2))$.

Now, suppose that $v_1 = u_1$ and $v_2 = h \circ u_2$ where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and concave and $h(0) = 0$ (player 2 is more risk averse).

Let $\langle S', d' \rangle$ be bargaining problem when the preferences of the players are represented by v_1 and v_2.

Let z_u be the solution of

$$\max_{0 \le z \le 1} u_1(z) u_2(1 - z),$$

and z_v the corresponding solution when $u_i = v_i$ for $i = 1, 2$.

Then,

$$f^N(S,d) = (u_1(z_u), u_2(1-z_u)) \text{ and } f^N(S',d') = (v_1(z_v), v_2(1-z_v)).$$

If u_i for $i = 1,2$ and h are differentiable then z_u and z_v are, in respect, the solutions of

$$\frac{u_1'(z)}{u_1(z)} = \frac{u_2'(1-z)}{u_2(1-z)}, \tag{1}$$

and

$$\frac{u_1'(z)}{u_1(z)} = \frac{h'(u_2(1-z))u_2'(1-z)}{h(u_2(1-z))}. \tag{2}$$

Since h is increasing and concave and $h(0) = 0$ we have

$$h'(t) \leq \frac{h(t)}{t}$$

for all t, so the RHS of (1) is at least as the RHS of (2) and thus $z_u \leq z_v$. Thus, if player 2 becomes more risk-averse, then f_1^N increases and f_2^N decreases.

If player 2's marginal utility declines more rapidly than that of player 1, then player 1's share exceeds $1/2$.

Monotonicity

Individual monotonicity (INM)

Let $\bar{s}_i$ be the maximum utility player i gets in $\{s \in S : s \geq d\}$.

(i) For any $\langle S, d \rangle$ and $\langle T, d \rangle$ with $S \subset T$ and $\bar{s}_i = \bar{t}_i$ for $i = 1, 2$, we have

$$f_i(S, d) \leq f_i(T, d)$$

for $i = 1, 2$.

(ii) For any $\langle S, d\rangle$ and $\langle T, d\rangle$ with $S \subset T$ and $\bar{s}_i = \bar{t}_i$ for i, we have

$$f_j(S, d) \leq f_j(T, d)$$

for $j \neq i$.

Strong monotonicity (STM): For any $\langle S, d\rangle$ and $\langle T, d\rangle$ with $S \subset T$, we have

$$f(S, d) \leq f(T, d).$$

Kalai-Smorodinsky

The unique bargaining solution

$$f^{KS} : B \rightarrow \mathbb{R}^2$$

satisfying SYM, WPO, INV and INM is given by

$$f^{KS}(S, d) = \{\frac{s_1}{\bar{s}_1} = \frac{s_2}{\bar{s}_2} : s \in S\} \cap WPO(S).$$

Proof

Normalize $(d_1, d_2) = (0, 0)$ and define

$$S' = \{(\frac{s_1}{\bar{s}_1}, \frac{s_2}{\bar{s}_2}) : (s_1, s_2) \in S\}$$

and note that $\bar{s}'_i = 1$ for each $i = 1, 2$.

By INV we have that

$$\frac{f_1^{KS}(S)}{\bar{s}_1} = f_1^{KS}(S') = f_2^{KS}(S') = \frac{f_2^{KS}(S)}{\bar{s}_2}.$$

Next, we show that f^{KS} is the only solution that satisfies SYM, WPO, INV and INM.

Let

$$T := co\{(0,0),(1,0),(0,1),f^{KS}(S')\}$$

and note that

$$f^{KS}(T) = f^{KS}(S')$$

and that for any $f(T)$ that satisfies WPO and SYM we have

$$f(T) = f^{KS}(T).$$

By INM we have that

$$f_i(S') \geq f_i(T) = f_i^{KS}(S').$$

for $i = 1, 2$.

By WPO of f^{KS} we know that $f(S') \leq f^{KS}(S')$ and thus

$$f(S') = f^{KS}(S').$$

And, by INV we have that

$$f(S) = f^{KS}(S)$$

which completes the proof.

Kalai

The unique bargaining solution

$$f^{KS} : B \rightarrow \mathbb{R}^2$$

satisfying SYM, WPO and STM is given by

$$f^K(S,d) = \max\{(s_1, s_2) \in S : s_1 = s_2\}.$$

Proof

Normalize $(d_1, d_2) = (0, 0)$ and define the symmetric set

$$\begin{aligned} T &= \{s \in S : (s_1, s_2) \in S \\ &\Leftrightarrow (s_2, s_1) \in S, s \leq f^K(S)\}. \end{aligned}$$

For example, the set T can be given by

$$T = \{s \in S : s_1 = s_2\}.$$

For any solution f that satisfies SYM and WPO

$$f(T) = f^K(S).$$

Since $T \subset S$, by STM, $f(T) \leq f(S)$ and thus $f^K(S) \leq f(S)$.

By WPO of $f^K(S) \geq f(S)$ so we have that

$$f^K(S) = f(S)$$

which concludes the proof.

The relation between the axiomatic and strategic approaches

To establish a common underlying model, we first introduce uncertainty into a bargaining game of alternating offers $\Gamma(\delta)$.

To this end, consider a game $\Gamma(q)$ with

– an exogenous possibility $0 < q < 1$ of breakdown B

– indifference about the period in which an agreement is reached.

A pair of strategies (σ, τ) that generates the outcome (x, t) in $\Gamma(\delta)$ leads to the outcome $\langle\langle x, t\rangle\rangle$ defined as

$$\begin{cases} x & \text{w/ prob. } (1-q)^t \\ B & \text{o/w} \end{cases}$$

in the game $\Gamma(q)$.

The key element in the analysis is the exact correspondence between $\Gamma(q)$ and $\Gamma(\delta)$so we can apply the Rubinstein's result to $\Gamma(q)$.

Assumptions on $\succsim_i$

[B1] Pie is desirable: for any $x, y \in X$, $x \succ_i y$ if and only if $x_i > y_i$ for $i = 1, 2$.

[B2] B is the worse outcome: $(0, 1) \sim_1 B$ and $(1, 0) \sim_2 B$.

[B3] Risk aversion: for any $x, y \in X$, and $(\alpha x + (1-\alpha)y) \in X$ for

$$(\alpha x + (1-\alpha)y) \succsim_i \alpha x + (1-\alpha)y \text{ for any } \alpha \in [0, 1].$$

Under **B1**-**B3**, $\succsim_i$ for $i = 1, 2$ over $\langle\langle x, t\rangle\rangle$ is complete, transitive, and

$$\langle\langle x, t\rangle\rangle \succ_i \langle\langle y, s\rangle\rangle \Leftrightarrow (1-q)^t u_i(x) > (1-q)^s u_i(y).$$

Under **B1** and **B3**, $u_i(x_i)$ is increasing and concave, and under **B2** we set (WLOG) $u_i(B) = 0$. Thus, $\langle S, d\rangle$ defined by

$$S = \{(s_1, s_2) \in \mathbb{R}^2_{++} : (s_1, s_2) = (u_1(x_1), u_2(x_2)) \text{ for some } x \in X\},$$

and

$$d = (u_1(B), u_2(B)) = (0, 0)$$

is a bargaining problem.

Next, we show that **B1**-**B3** ensure that $\succsim_i$ over $\langle\langle x, t\rangle\rangle$ (lotteries) satisfy assumptions **A1**-**A6** when we replace the symbol (x, t) with $\langle\langle x, t\rangle\rangle$.

[A1] $\langle\langle x, t\rangle\rangle \succsim_i B$ for all outcomes $\langle\langle x, t\rangle\rangle$ and $i = 1, 2$.

[A2] $\langle\langle x, t\rangle\rangle \succ_i \langle\langle y, t\rangle\rangle$ for $x_i > y_i$ and $i = 1, 2$

[A3] $\langle\langle x, t\rangle\rangle \succsim_i \langle\langle x, s\rangle\rangle$ for $t < s$ and $i = 1, 2$, and the preferences are strict if $x_i > 0$.

[A4] trivial (by the continuity of each u_i).

[A5] trivial (by the continuity of each u_i).

[A6] The concavity of u_i implies that if $x_i < y_i$ then

$$\frac{u_i(x_i) - u_i(v_i(x_i, 1))}{x_i - v_i(x_i, 1)} \geq \frac{u_i(y_i) - u_i(v_i(y_i, 1))}{y_i - v_i(y_i, 1)}$$

and since $u_i(v_i(x_i, 1)) = (1 - q)u_i(x_i)$, we get

$$\frac{qu_i(x_i)}{x_i - v_i(x_i, 1)} \geq \frac{qu_i(y_i)}{y_i - v_i(y_i, 1)}.$$

Since $u_i(x_i) < u_i(y_i)$ then $x_i - v_i(x_i, 1) < y_i - v_i(y_i, 1)$.

Subgame perfect equilibrium of $\Gamma(q)$

For each $q \in (0,1)$, the pair of agreements $(x^*(q), y^*(q))$ satisfying

$$\langle\langle y^*(q), 0\rangle\rangle \sim_1 \langle\langle x^*(q), 1\rangle\rangle \text{ and } \langle\langle x^*(q), 0\rangle\rangle \sim_2 \langle\langle y^*(q), 1\rangle\rangle$$

or, equivalently

$$u_1(y^*(q)) = (1-q)u_1(x^*(q)) \text{ and } u_2(x^*(q)) = (1-q)u_2(y^*(q))$$

is the unique SPE of $\Gamma(q)$ (by the Rubinstein's result above).

In the unique SPE, player 1proposes the agreement $x^*(q)$ in period 0 and player 2 accepts.

As q approaches 0 the agreement $x^*(q)$ reached in the unique SPE of $\Gamma(q)$ approaches the agreement given by $f^N(S,d) = \arg\max u_1(x_1)u_2(x_2)$ where

$$S = \{(s_1, s_2) \in \mathbb{R}^2_{++} : (s_1, s_2) = (u_1(x_1), u_2(x_2)) \text{ for some } x \in X\},$$

and

$$d = (u_1(B), u_2(B)) = (0, 0).$$

From the equilibrium condition

$$u_1(x^*(q))u_2(x^*(q)) = u_1(y^*(q))u_2(y^*(q))$$

and that

$$\lim_{q \to 0}[u_i(x^*(q)) - u_i(y^*(q)] = 0$$

for $i = 1, 2$.

Nash's (1953) demand game

Nash (1953) proposed a strategic game of bargaining that "supports" his axiomatic solution.

Nash's demand game is the two-player strategic game G where for $i = 1, 2$

- $A_i = [0, \infty)$
- $h_i : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by

$$h_i(\sigma_1, \sigma_2) = \begin{cases} 0 & \text{if} \quad (\sigma_1, \sigma_2) \notin S \\ \sigma_i & \text{if} \quad (\sigma_1, \sigma_2) \in S \end{cases}$$

where σ_i is the "demand" of player i.

Any strategy pair (σ_1, σ_2) which is SPO is a Nash equilibrium of the game.

In a perturbed demand game, there is some uncertainty in the neighborhood of the boundary of S.

The main idea is to requires that an equilibrium will be robust to perturbations in the structure of the game (Selten, 1975).

Specifically, any demand-pair (σ_1, σ_2) results an agreement (σ_1, σ_2) with probability $P(\sigma_1, \sigma_2)$ where

$$P(\sigma_1, \sigma_2) = \begin{cases} 0 & \text{if } (\sigma_1, \sigma_2) \notin S \\ [0,1] & \text{if } (\sigma_1, \sigma_2) \in S \end{cases}$$

and $0 < P(\sigma_1, \sigma_2)$ if (σ_1, σ_2) is in the interior of S.

Further, suppose $P : \mathbb{R}^2_+ \to [0,1]$ is differentiable and quasi-concave, so for each $\rho \in [0,1]$ the set

$$\bar{P}(\rho) = \{(\sigma_1, \sigma_2) \in \mathbb{R}^2_+ : P(\sigma_1, \sigma_2) \geq \rho\}$$

is convex.

Let G^n be the perturbed demand game defined by $\langle S, d \rangle$ and P^n and assume that the (Hausdorff) distance between S and $\overline{P^n}(1)$ associated with P^n converges to 0 as $n \longrightarrow \infty$.

Nash 1953: [1] Every game G^n has a Nash equilibrium in which an agreement is reached with positive probability. [2] The limit as $n \longrightarrow \infty$ for every sequence $\{\sigma^{*n}\}_{n=1}^{\infty}$ in which σ^{*n} is such a Nash equilibrium is the Nash solution $f^N(S, d)$.

Proof: Let

$$(\hat{\sigma}_1, \hat{\sigma}_2) \in \underset{(\sigma_1,\sigma_2)\in\mathbb{R}^2_+}{\text{arg max}} \; \sigma_1\sigma_2 P^n(\sigma_1, \sigma_2).$$

Since P^n is continuous and $P^n(\sigma_1, \sigma_2) = 0$ if $(\sigma_1, \sigma_2) \notin S$ then $(\hat{\sigma}_1, \hat{\sigma}_2)$ exists; and since $\hat{\sigma}_i > 0$ for both $i = 1, 2$ then $P^n(\hat{\sigma}_1, \hat{\sigma}_2) > 0$.

Further, since

$$\hat{\sigma}_1 \in \underset{\sigma_1\in\mathbb{R}_+}{\text{arg max}} \sigma_1 P^n(\sigma_1, \hat{\sigma}_2) \text{ and } \hat{\sigma}_2 \in \underset{\sigma_2\in\mathbb{R}_+}{\text{arg max}} \sigma_2 P^n(\hat{\sigma}_1, \sigma_2)$$

then $(\hat{\sigma}_1, \hat{\sigma}_2)$ is a Nash equilibrium.

Let $(\sigma_1^*, \sigma_2^*) \in S$ be an equilibrium of G^n where $\sigma_i^* > 0$ for all $i = 1, 2$.

Since P^n is differentiable and $\sigma_i^* = BR(\sigma_j^*)$ then

$$\sigma_i^* D_i P^n(\sigma^*) + P^n(\sigma^*) = 0 \text{ for } i = 1, 2$$

(where D_i is the partial derivative of P^n with respect to its i^{th} argument) and thus

$$\frac{D_1 P^n(\sigma^*)}{D_2 P^n(\sigma^*)} = \frac{\sigma_1^*}{\sigma_2^*}. \qquad (\star)$$

Let $\pi^* = P^n(\sigma_1^*, \sigma_2^*)$, so that

$$(\sigma_1^*, \sigma_2^*) \in \overline{P^n}(\pi^*) \cap WPO$$

(since it is an equilibrium).

Since P^n is quasi-concave, $(\star)$ implies that σ_1^*, σ_2^* is the maximizer of $\sigma_1\sigma_2$ subject to $P^n(\sigma_1, \sigma_2) \geq \pi^*$, in particular

$$\sigma_1^*\sigma_2^* \geq \max_{(\sigma_1,\sigma_2)} \{\sigma_1\sigma_2 : (\sigma_1, \sigma_2) \in \overline{P^n}(1)\}$$

which completes the proof.