Microeconomics III

Nash equilibrium I
(Mar 18, 2012)

School of Economics
The Interdisciplinary Center (IDC), Herzliya

Terminology and notations

Preferences $\succsim$ is a binary relation on some set of alternatives A. From $\succsim$ we derive two other relations on A:

- strict performance relation

$$a \succ b \iff a \succsim b \text{ and not } b \succsim a$$

- indifference relation

$$a \sim b \iff a \succsim b \text{ and } b \succsim a$$

Utility representation $\succsim$ is said to be

- complete if $\forall a, b \in A$, $a \succsim b$ or $b \succsim a$.
- transitive if $\forall a, b, c \in A$, $a \succsim b$ and $b \succsim c$ then $a \succsim c$.

$\succsim$ can be presented by a utility function only if it is complete and transitive (rational).

A function $u : A \to \mathbb{R}$ is a utility function representing $\succsim$ if $\forall a, b \in A$

$$a \succsim b \iff u(a) \geq u(b).$$

Profiles Let N be a the set of players.

- $(a_i)_{i \in N}$ or simply (a_i) is an action profile — a collection actions, one for each player.

- $(a_j)_{j \in N/\{i\}}$ or simply a_{-i} is the list of elements of the action profile $(a_j)_{j \in N}$ for all players except for player i.

- (a_i, a_{-i}) is the action a_i and the list of actions a_{-i}, which is the action profile $(a_i)_{i \in N}$.

Games and solutions

A game - a model of interactive (multi-person) decision-making. We distinguish between:

– Noncooperative and cooperative games — the units of analysis are individuals or (sub) groups.

– Strategic (normal) form games and extensive form games — players move simultaneously or precede one another.

– Gams with perfect and imperfect information — players are perfectly or imperfectly informed about characteristics, events and actions.

A solution - a systematic description of outcomes in a family of games.

– Nash equilibrium — strategic form games.

– Subgame perfect equilibrium — extensive form games with perfect information.

– Perfect Bayesian equilibrium — games with observable actions.

– Sequential equilibrium (and refinements) — extensive form games with imperfect information.

Formalities

A strategic game A *finite* set N of players, and for each player $i \in N$

- a non-empty set A_i of actions
- a preference relation $\succsim_i$on the set $A = A_1 \times A_2 \times \cdots \times A_N$ of possible outcomes.

We will denote a strategic game by

$$\langle N, (A_i), (\succsim_i) \rangle$$

or by

$$\langle N, (A_i), (u_i) \rangle$$

when $\succsim_i$can be represented by a utility function $u_i : A \to \mathbb{R}$.

A two-player finite strategic game can be described conveniently in a bi-matrix.

For example, a 2×2 game

	L	R
T	A_1, A_2	B_1, B_2
B	C_1, C_2	D_1, D_2

Best response

For any list of strategies $a_{-i} \in A_{-i}$

$$B_i(a_{-i}) = \{a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a_i') \ \forall a_i' \in A_i\}$$

is the set of players i's best actions given a_{-i}.

Strategy a_i is i's best response to a_{-i} if it is the optimal choice when i conjectures that others will play a_{-i}.

Nash equilibrium

Nash equilibrium (NE) is a steady state of the play of a strategic game.

A NE of a strategic game $\langle N, (A_i), (\succsim_i)\rangle$ is a profile $a^* \in A$ of actions such that

$$(a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a_i)$$

$\forall a_i \in A_i$ and $\forall i \in N$, or equivalently

$$a^*_i \in B_i(a^*_{-i})$$

$\forall i \in N$.

In words, no player has a profitable deviation given the actions of the other players.

Classical 2×2 **games**

Prisoner's Dilemma

	L	R
T	$3, 3$	$0, 4$
B	$4, 0$	$1, 1$

BoS

	L	R
T	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Coordination

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

Hawk-Dove

	L	R
T	$3, 3$	$0, 4$
B	$4, 0$	$1, 1$

Matching Pennies

	L	R
T	$1, -1$	$-1, 1$
B	$-1, 1$	$1, 1$

Existence of Nash equilibrium

Let the set-valued function $B : A \longrightarrow A$ defined by

$$B(a) = \times_{i \in N} B_i(a_{-i})$$

and rewrite the equilibrium condition

$$a_i^* \in B_i(a_{-i}^*) \ \forall i \in N$$

in vector form as follows

$$a^* \in B(a^*)$$

Kakutani's fixed point theorem gives conditions on B under which $\exists a^*$ such that $a^* \in B(a^*)$.

Kakutani's fixed point theorem

Let $X \subseteq \mathbb{R}^n$ be non-empty compact (closed and bounded) and convex set and $f : X \longrightarrow X$ be a set-valued function for which

- the set $f(x)$ is non-empty and convex $\forall x \in X$.
- the graph of f is closed

  $y \in f(x)$ for any $\{x_n\}$ and $\{y_n\}$ such that
  $y_n \in f(x_n) \forall n$ and $x_n \longrightarrow x$ and $y_n \longrightarrow y$.

Than, $\exists x^* \in X$ such that $x^* \in f(x^*)$.

Necessity of conditions in Kakutani's theorem

– X is compact

$$X = \mathbb{R}^1 \text{ and } f(x) = x + 1$$

– X is convex

$$X = \{x \in \mathbb{R}^2 : \|x\| = 1\} \text{ and } f \text{ is } 90^\circ \text{ clock-wise rotation.}$$

- $f(x)$ is convex for any $x \in X$

  $X = [0, 1]$ and

$$f(x) = \begin{cases} \{1\} & \text{if} \quad x < \frac{1}{2}, \\ \{0, 1\} & \text{if} \quad x = \frac{1}{2}, \\ \{0\} & \text{if} \quad x > \frac{1}{2}. \end{cases}$$

- f has a closed graph

  $X = [0, 1]$ and

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 1, \\ 0 & \text{if} \quad x = 1. \end{cases}$$

A strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a NE if for all $i \in N$

- A_i is non-empty, compact and convex.
- $\succsim_i$ is continuous and quasi-concave on A_i.

B has a fixed point by Kakutani:

- $B_i(a_{-i}) \neq \emptyset$ (A_i is compact and $\succsim_i$ is continuous).
- $B_i(a_{-i})$ is convex ($\succsim_i$is quasi-concave on A_i).
- B has a closed graph ($\succsim_i$is continuous).

Dominance

An action $a_i' \in A_i$ of player i is strictly dominated if there exists another action a_i'' such that

$$u_i(a_i', a_{-i}) < u_i(a_i'', a_{-i})$$

for all $a_{-i} \in A_{-i}$.

An action $a_i' \in A_i$ of player i is weakly dominated if there exists another action a_i'' such that

$$u_i(a_i', a_{-i}) \leq u_i(a_i'', a_{-i})$$

for all $a_{-i} \in A_{-i}$ and the inequality is strict for some $a_{-i} \in A_{-i}$.