Microeconomics III

Bargaining I
The strategic approach
(May 13, 2012)

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The strategic approach

The players bargain over a pie of size 1.

An agreement is a pair (x_1, x_2) where x_i is player i's share of the pie. The set of possible agreements is

$$X = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$$

Player i prefers $x \in X$ to $y \in X$ if and only if $x_i > y_i$.

The bargaining protocol

The players can take actions only at times in the (infinite) set $T = \{0, 1, 2, ...\}$. In each $t \in T$ player i, proposes an agreement $x \in X$ and $j \neq i$ either accepts (Y) or rejects (N).

If x is accepted (Y) then the bargaining ends and x is implemented. If x is rejected (N) then the play passes to period $t + 1$ in which j proposes an agreement.

At all times players have perfect information. Every path in which all offers are rejected is denoted as disagreement (D). The only asymmetry is that player 1 is the first to make an offer.

Preferences

Time preferences (toward agreements at different points in time) are the driving force of the model.

A bargaining game of alternating offers is

- an extensive game of perfect information with the structure given above, and

- player i's preference ordering $\precsim_i$ over $(X \times T) \cup \{D\}$ is complete and transitive.

Preferences over $X \times T$ are represented by $\delta_i^t u_i(x_i)$ for any $0 < \delta_i < 1$ where u_i is increasing and concave.

Assumptions on preferences

A1 Disagreement is the worst outcome

For any $(x, t) \in X \times T$,

$$(x, t) \succsim_i D$$

for each i.

A2 Pie is desirable

– For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t) \text{ if and only if } x_i > y_i.$$

A3 Time is valuable

For any $t \in T$, $s \in T$ and $x \in X$

$$(x, t) \succsim_i (x, s) \text{ if } t < s$$

and with strict preferences if $x_i > 0$.

A4 Preference ordering is continuous

Let $\{(x_n, t)\}_{n=1}^{\infty}$ and $\{(y_n, s)\}_{n=1}^{\infty}$ be members of $X \times T$ for which

$$\lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y.$$

Then, $(x, t) \succsim_i (y, s)$ whenever $(x_n, t) \succsim_i (y_n, s)$ for all n.

A2-A4 imply that for any outcome (x, t) either there is a unique $y \in X$ such that

$$(y, 0) \sim_i (x, t)$$

or

$$(y, 0) \succ_i (x, t)$$

for every $y \in X$.

Note $\succsim_i$ satisfies **A2-A4** iff it can be represented by a continuous function

$$U_i : [0, 1] \times T \longrightarrow \mathbb{R}$$

that is increasing (deceasing) in the first (second) argument.

A5 Stationarity

For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t+1) \text{ if and only if } (x, 0) \succ_i (y, 1).$$

If $\succsim_i$ satisfies **A2-A5** then for every $\delta \in (0,1)$ there exists a continuous increasing function $u_i : [0,1] \rightarrow \mathbb{R}$ (not necessarily concave) such that

$$U_i(x_i, t) = \delta_i^t u_i(x_i).$$

Present value

Define $v_i : [0,1] \times T \to [0,1]$ for $i = 1, 2$ as follows

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y, 0) \sim_i (x, t) \\ 0 & \text{if } (y, 0) \succ_i (x, t) \text{ for all } y \in X. \end{cases}$$

We call $v_i(x_i, t)$ player i's present value of (x, t) and note that

$$(y, t) \succ_i (x, s) \text{ whenever } v_i(y_i, t) > v_i(x_i, s).$$

If $\succsim_i$satisfies **A2**-**A4**, then for any $t \in T$ $v_i(\cdot, t)$ is continuous, non decreasing and increasing whenever $v_i(x_i, t) > 0$.

Further, $v_i(x_i, t) \leq x_i$ for every $(x, t) \in X \times T$ and with strict whenever $x_i > 0$ and $t \geq 1$.

With **A5**, we also have that

$$v_i(v_i(x_i, 1), 1) = v_i(x_i, 2)$$

for any $x \in X$.

Delay

A6 Increasing loss to delay

$x_i - v_i(x_i, 1)$ is an increasing function of x_i.

If u_i is differentiable then under **A6** in any representation $\delta_i^t u_i(x_i)$ of $\succsim_i$

$$\delta_i u_i'(x_i) < u_i'(v_i(x_i, 1))$$

whenever $v_i(x_i, 1) > 0$.

This assumption is weaker than concavity of u_i which implies

$$u_i'(x_i) < u_i'(v_i(x_i, 1)).$$

The single crossing property of present values

If $\succsim_i$for each i satisfies **A2-A6**, then there exist a unique pair $(x^*, y^*) \in X \times X$ such that

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

– For every $x \in X$, let $\psi(x)$ be the agreement for which

$$\psi_1(x) = v_1(x_1, 1)$$

and define $H : X \to \mathbb{R}$ by

$$H(x) = x_2 - v_2(\psi_2(x), 1).$$

– The pair of agreements x and $y = \psi(x)$ satisfies also $x_2 = v_2(\psi_2(x), 1)$ iff $H(x) = 0$.

– Note that $H(0, 1) \geq 0$ and $H(1, 0) \leq 0$, H is a continuous function, and

$$
\begin{aligned}
H(x) &= [v_1(x_1, 1) - x_1] + \\
&\quad +[1 - v_1(x_1, 1) - v_2(1 - v_1(x_1, 1), 1)].
\end{aligned}
$$

– Since $v_1(x_1, 1)$ is non decreasing in x_1, and both terms are decreasing in x_1, H has a unique zero by **A6**.

Examples

[1] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = \delta_i^t x_i$$

where $\delta_i \in (0, 1)$, and $U_i(D) = 0$.

[2] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = x_i - c_i t$$

where $c_i > 0$, and $U_i(D) = -\infty$ (constant cost of delay).

Although **A6** is violated, when $c_1 \neq c_2$ there is a unique pair $(x, y) \in X \times X$ such that $y_1 = v_1(x_1, 1)$ and $x_2 = v_2(y_2, 1)$.

Strategies

Let X^t be the set of all sequences $\{x^0, ..., x^{t-1}\}$ of members of X.

A strategy of player 1 (2) is a sequence of functions

$$\sigma = \{\sigma^t\}_{t=0}^{\infty}$$

such that $\sigma^t : X^t \to X$ if t is even (odd), and $\sigma^t : X^{t+1} \to \{Y, N\}$ if t is odd (even).

The way of representing a player's strategy in closely related to the notion of automation.

Nash equilibrium

For any $\bar{x} \in X$, the outcome $(\bar{x}, 0)$ is a NE when players' preference satisfy **A1**-**A6**.

To see this, consider the stationary strategy profile

Player 1	proposes	$\bar{x}$
	accepts	$x_1 \geq \bar{x}_1$
Player 2	proposes	$\bar{x}$
	accepts	$x_2 \geq \bar{x}_2$

This is an example for a pair of one-state automate.

The set of outcomes generated in the Nash equilibrium includes also delays (agreements in period 1 or later).

Subgame perfect equilibrium

Any bargaining game of alternating offers in which players' preferences satisfy **A1**-**A6** has a unique SPE which is the solution of the following equations

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

Note that if $y_1^* > 0$ and $x_2^* > 0$ then

$$(y_1^*, 0) \sim_1 (x_1^*, 1) \text{ and } (x_2^*, 0) \sim_2 (y_2^*, 1).$$

The equilibrium strategy profile is given by

Player 1	proposes	x^*
	accepts	$y_1 \geq y_1^*$
Player 2	proposes	y^*
	accepts	$x_2 \geq x_2^*$

The unique outcome is that player 1 proposes x^* in period 0 and player 2 accepts.

Step 1 (x^*, y^*) is a SPE

Player 1:

- proposing x^* at t^* leads to an outcome (x^*, t^*). Any other strategy generates either

  $$(x, t) \text{ where } x_1 \leq x_1^* \text{ and } t \geq t^*$$

  or

  $$(y^*, t) \text{ where } t \geq t^* + 1$$

  or D.

- Since $x_1^* > y_1^*$ it follows from **A1**-**A3** that (x^*, t^*) is a best response.

Player 2:

– accepting x^* at t^* leads to an outcome (x^*, t^*). Any other strategy generates either

$$(y, t) \text{ where } y_2 \leq y_2^* \text{ and } t \geq t^* + 1$$

or

$$(x^*, t) \text{ where } t \geq t^*$$

or D.

– By **A1**-**A3** and **A5**

$$(x^*, t^*) \succsim_2 (y^*, t^* + 1)$$

and thus accepting x^* at t^*, which leads to the outcome (x^*, t^*), is a best response.

Note that similar arguments apply to a subgame starting with an offer of player 2.

Step 2 (x^*, y^*) is the unique SPE

Let G_i be a subgame starting with an offer of player i and define

$$M_i = \sup\{v_i(x_i, t) : (x, t) \in SPE(G_i)\},$$

and

$$m_i = \inf\{v_i(x_i, t) : (x, t) \in SPE(G_i)\}.$$

It is suffices to show that

$$M_1 = m_1 = x_1^* \text{ and } M_2 = m_2 = y_2^*.$$

First, note that in any SPE the first offer is accepted because

$$v_1(y_1^*, 1) \leq y_1^* < x_1^*.$$

Thus, after a rejection, the present value for player 1 is less than x_1^*.

Then, it remains to show that

$$m_2 \geq 1 - v_1(M_1, 1) \tag{1}$$

and

$$M_1 \leq 1 - v_2(m_2, 1). \tag{2}$$

1 implies that the pair $(M_1, 1 - m_2)$ lies below the line

$$y_1 = v_1(x_1, 1)$$

and 2 implies that the pair $(M_1, 1 - m_2)$ lies to the left the line

$$x_2 = v_2(y_2, 1).$$

Thus,

$$M_1 = x_1^* \text{ and } m_2 = y_2^*,$$

and with the role of the players reversed, the same argument show that

$$M_2 = y_2^* \text{ and } m_1 = x_1^*.$$

With constant discount rates the equilibrium condition implies that

$$y_1^* = \delta_1 x_1^* \text{ and } x_2^* = \delta_2 y_2^*$$

so that

$$x^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} \right) \text{ and } y^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2} \right).$$

Thus, if $\delta_1 = \delta_2 = \delta$ $(v_1 = v_2)$ then

$$x^* = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) \text{ and } y^* = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$$

so player 1 obtains more than half of the pie.

But, shrinking the length of a period by considering a sequence of games indexed by Δ in which $u_i = \delta_i^{\Delta t} x_i$ we have

$$\lim_{\Delta \to 0} x^*(\Delta) = \lim_{\Delta \to 0} y^*(\Delta) = \left(\frac{\log \delta_2}{\log \delta_1 + \log \delta_2}, \frac{\log \delta_1}{\log \delta_1 + \log \delta_2}\right).$$