UC Berkeley
Haas School of Business
Game Theory
(EMBA 296 & EWMBA 211)
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Advanced Topics:
Evolutionary Game Theory, Repeated Games, Auctions, Bargaining

Block 4

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Evolutionary Game Theory

Evolutionary stability

A single population of players. Players interact with each other pair-wise and randomly matched.

Players are assigned modes of behavior (mutation). Utility measures each player's ability to survive.

ε of players consists of mutants taking action a while others take action a^*.

Evolutionary stable strategy (ESS)

Consider a two-player payoff symmetric game

$$G = \langle \{1, 2\}, (A, A), (u_1, u_2) \rangle$$

where

$$u_1(a_1, a_2) = u_2(a_2, a_1)$$

(players exchanging a_1 and a_2).

$a^* \in A$ is ESS if and only if for any $a \in A$, $a \neq a^*$ and $\varepsilon > 0$ sufficiently small

$$(1-\varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) > (1-\varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

which is satisfied if and only if for any $a \neq a^*$ either

$$u(a^*, a^*) > u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) > u(a, a)$$

Three results on ESS

[1] If a^* is an ESS then (a^*, a^*) is a NE.

Suppose not. Then, there exists a strategy $a \in A$ such that

$$u(a, a^*) > u(a^*, a^*).$$

But, for ε small enough

$$(1 - \varepsilon)u(a^*, a^*) + \varepsilon u(a^*, a) < (1 - \varepsilon)u(a, a^*) + \varepsilon u(a, a)$$

and thus a^* is not an ESS.

[2] If (a^*, a^*) is a strict NE $(u(a^*, a^*) > u(a, a^*)$ for all $a \in A)$ then a^* is an ESS.

Suppose a^* is not an ESS. Then either

$$u(a^*, a^*) \leq u(a, a^*)$$

or

$$u(a^*, a^*) = u(a, a^*) \text{ and } u(a^*, a) \leq u(a, a).$$

so (a^*, a^*) can be a NE but not a strict NE.

[3] The two-player two-action game

	a	a'
a	w, w	x, y
a'	y, x	z, z

has a strategy which is ESS.

If $w > y$ or $z > x$ then (a, a) or (a', a') are strict NE, and thus a or a' are ESS.

If $w < y$ and $z < x$ then there is a unique symmetric mixed strategy NE (α^*, α^*) where

$$\alpha^*(a) = (z - x)/(w - y + z - x)$$

and $u(\alpha^*, \alpha) > u(\alpha, \alpha)$ for any $\alpha \neq \alpha^*$.

Repeated games
(the prisoner's dilemma)

The basic idea – prisoner's dilemma

In the Prisoner's Dilemma

	C	D
C	3, 3	0, 4
D	4, 0	1, 1

No cooperation (D, D) is the unique NE since D strictly dominates C, but both players are better off when the outcome is (C, C).

When played repeatedly, cooperation (C, C) in every period is stable if

– each player believes that choosing D will end cooperation, and

– subsequent losses outweigh the immediate gain.

The socially desirable outcome (C, C) can be sustained if (and only if) players have long-term objectives.

In general, we can think that strategies are social norms, cooperation, threats and punishments where threats are carried out as punishments when the social norms require it.

Strategies

Grim trigger strategy

$$\boxed{\mathcal{C} : C} \;\underset{(\cdot, D)}{\longrightarrow}\; \boxed{\mathcal{D} : D}$$

Limited punishment

$$\dashrightarrow \boxed{\mathcal{P}_0 : C} \;\underset{(\cdot, D)}{\longrightarrow}\; \boxed{\mathcal{P}_1 : D} \;\underset{(\cdot, \cdot)}{\longrightarrow}\; \boxed{\mathcal{P}_2 : D} \;\underset{(\cdot, \cdot)}{\longrightarrow}\; \boxed{\mathcal{P}_3 : D} \;\underset{(\cdot, \cdot)}{\dashrightarrow}$$

Tit-for-tat

$$\dashrightarrow \boxed{\mathcal{C} : C} \;\underset{(\cdot, D)}{\longrightarrow}\; \boxed{\mathcal{D} : D} \;\underset{(\cdot, C)}{\dashrightarrow}$$

Payoffs

A player's preferences over an infinite stream $(\omega^1, \omega^2, ...)$ of payoffs are represented by the discounted sum

$$V = \sum_{t=1}^{\infty} \delta^{t-1} \omega^t,$$

where $0 < \delta < 1$.

The discounted sum of stream $(c, c, ...)$ is $\dfrac{C}{1-\delta}$, so a player is indifferent between the two streams if

$$c = (1 - \delta)V.$$

Hence, we call $(1 - \delta)V$ the discounted average of stream $(\omega^1, \omega^2, ...)$, which represent the same preferences.

To elucidate, let

$$S_T = c + \delta c + \delta^2 c + \cdots + \delta^T c$$

and note that

$$\delta S_T = \delta c + \delta^2 c + \delta^3 c + \cdots + \delta^{T+1} c$$

so that $S_T - \delta S_T = c - \delta^{T+1} c$ and thus

$$S_T = \frac{1 - \delta^{T+1}}{1 - \delta} c,$$

which equals $\dfrac{C}{1 - \delta}$ as $T \to \infty$.

Nash equilibria

<u>Grim trigger strategy</u>

$$(1-\delta)(3+\delta+\delta^2+\cdots) = (1-\delta)\left[3+\frac{\delta}{(1-\delta)}\right] = 3(1-\delta)+\delta$$

Thus, a player cannot increase her payoff by deviating if and only if

$$3(1-\delta)+\delta \leq 2,$$

or $\delta \geq 1/2$.

If $\delta \geq 1/2$, then the strategy pair in which each player's strategy is grim strategy is a Nash equilibrium which generates the outcome (C, C) in every period.

Limited punishment (k periods)

$$(1-\delta)(3+\delta+\delta^2+\cdots+\delta^k) = (1-\delta)\left[3+\delta\frac{(1-\delta^k)}{(1-\delta)}\right] = 3(1-\delta)+\delta(1-\delta^k)$$

Note that after deviating at period t a player should choose D from period $t+1$ through $t+k$.

Thus, a player cannot increase her payoff by deviating if and only if

$$3(1-\delta)+\delta(1-\delta^k) \leq 2(1-\delta^{k+1}).$$

Note that for $k=1$, then no $\delta < 1$ satisfies the inequality.

Tit-for-tat

A deviator's best-reply to tit-for-tat is to alternate between D and C or to always choose D, so tit-for tat is a best-reply to tit-for-tat if and only if

$$(1-\delta)(3+0+3\delta^2+0+\cdots)=(1-\delta)\frac{3}{1-\delta^2}=\frac{3}{1+\delta}\leq 2$$

and

$$(1-\delta)(3+\delta+\delta^2+\cdots)=(1-\delta)\left[3+\frac{\delta}{(1-\delta)}\right]=3-2\delta\leq 2.$$

Both conditions yield $\delta \geq 1/2$.

Auctions

Auction design

Two important issues for auction design are:

- Attracting entry
- Preventing collusion

Sealed-bid auction deals better with these issues, but it is more likely to lead to inefficient outcomes.

European 3G mobile telecommunication license auctions

Although the blocks of spectrum sold were very similar across countries, there was an enormous variation in revenues (in USD) per capita:

Austria	100
Belgium	45
Denmark	95
Germany	615
Greece	45
Italy	240
Netherlands	170
Switzerland	20
United Kingdom	650

United Kingdom

- 4 licenses to be auctioned off and 4 incumbents (with advantages in terms of costs and brand).

- To attract entry and deter collusion – an English until 5 bidders remain and then a sealed-bid with reserve price given by lowest bid in the English.

- later a 5th license became available to auction, a straightforward English auction was implemented.

Netherlands

- Followed UK and used (only) an English auction, but they had 5 incumbents and 5 licenses!

- Low participation: strongest potential entrants made deals with incumbents, and weak entrants either stayed out or quit bidding.

Switzerland

– Also followed UK and ran an English auction for 4 licenses. Companies either stayed out or quit bidding.

1. The government permitted last-minute joint-bidding agreements. Demand shrank from 9 to 4 bidders one week before the auction.

2. The reserve price had been set too low. The government tried to change the rules but was opposed by remaining bidders and legally obliged to stick to the original rules.

– Collected 1/30 per capita of UK, and 1/50 of what they had hoped for!

Types of auctions

Sequential / simultaneous

Bids may be called out sequentially or may be submitted simultaneously in sealed envelopes:

- English (or oral) – the seller actively solicits progressively higher bids and the item is soled to the highest bidder.
- Dutch – the seller begins by offering units at a "high" price and reduces it until all units are soled.
- Sealed-bid – all bids are made simultaneously, and the item is sold to the highest bidder.

First-price / second-price

The price paid may be the highest bid or some other price:

- First-price – the bidder who submits the highest bid wins and pay a price equal to her bid.

- Second-prices – the bidder who submits the highest bid wins and pay a price equal to the second highest bid.

Variants: all-pay (lobbying), discriminatory, uniform, Vickrey (William Vickrey, Nobel Laureate 1996), and more.

Private-value / common-value

Bidders can be certain or uncertain about each other's valuation:

- In private-value auctions, valuations differ among bidders, and each bidder is certain of her own valuation and can be certain or uncertain of every other bidder's valuation.

- In common-value auctions, all bidders have the same valuation, but bidders do not know this value precisely and their estimates of it vary.

First-price auction (with perfect information)

To define the game precisely, denote by v_i the value that bidder i attaches to the object. If she obtains the object at price p then her payoff is $v_i - p$.

Assume that bidders' valuations are all different and all positive. Number the bidders 1 through n in such a way that

$$v_1 > v_2 > \cdots > v_n > 0.$$

Each bidder i submits a (sealed) bid b_i. If bidder i obtains the object, she receives a payoff $v_i - b_i$. Otherwise, her payoff is zero.

Tie-breaking – if two or more bidders are in a tie for the highest bid, the winner is the bidder with the highest valuation.

In summary, a first-price sealed-bid auction with perfect information is the following strategic game:

- Players: the n bidders.
- Actions: the set of possible bids b_i of each player i (nonnegative numbers).
- Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \bar{b} & \text{if} \quad b_i = \bar{b} \text{ and } v_i > v_j \text{ if } b_j = \bar{b} \\ 0 & \text{if} \quad b_i < \bar{b} \end{cases}$$

where $\bar{b}$ is the highest bid.

The set of Nash equilibria is the set of profiles $(b_1, ..., b_n)$ of bids with the following properties:

$$
\begin{array}{ll}
[1] & v_2 \leq b_1 \leq v_1 \\
[2] & b_j \leq b_1 \ \underline{\text{for all}}\ j \neq 1 \\
[3] & b_j = b_1 \ \underline{\text{for some}}\ j \neq 1
\end{array}
$$

It is easy to verify that all these profiles are Nash equilibria. It is harder to show that there are no other equilibria. We can easily argue, however, that there is no equilibrium in which player 1 does not obtain the object.

$\Longrightarrow$ The first-price sealed-bid auction is socially efficient, but does not necessarily raise the most revenues.

Second-price auction (with perfect information)

A second-price sealed-bid auction with perfect information is the following strategic game:

- Players: the n bidders.
- Actions: the set of possible bids b_i of each player i (nonnegative numbers).
- Payoffs: the preferences of player i are given by

$$u_i = \begin{cases} v_i - \overline{b} & \text{if} \quad b_i > \overline{b} \text{ or } b_i = \overline{b} \text{ and } v_i > v_j \text{ if } b_j = \overline{b} \\ 0 & \text{if} \quad b_i < \overline{b} \end{cases}$$

  where $\overline{b}$ is the highest bid submitted by a player other than i.

First note that for any player i the bid $b_i = v_i$ is a (weakly) dominant action (a "truthful" bid), in contrast to the first-price auction.

The second-price auction has many equilibria, but the equilibrium $b_i = v_i$ for all i is distinguished by the fact that every player's action dominates all other actions.

Another equilibrium in which player $j \neq 1$ obtains the good is that in which

$$
\begin{array}{ll}
[1] & b_1 < v_j \text{ and } b_j > v_1 \\
[2] & b_i = 0 \ \underline{\text{for all}}\ i \neq \{1, j\}
\end{array}
$$

Common-value auctions and the winner's curse

Suppose we all participate in a sealed-bid auction for a jar of coins. Once you have estimated the amount of money in the jar, what are your bidding strategies in first- and second-price auctions?

The winning bidder is likely to be the bidder with the largest positive error (the largest overestimate).

In this case, the winner has fallen prey to the so-called the winner's curse. Auctions where the winner's curse is significant are oil fields, spectrum auctions, pay per click, and more.

The winner's curse has also been shown in stock market and real estate investments, mergers and acquisitions, and bidding on baseball players.

When Goggle launched its IPO by auction in 2004, the SEC registration statement said:

> "The auction process for our public offering may result in a phenomenon known as the 'winner's curse,' and, as a result, investors may experience significant losses (...) Successful bidders may conclude that they paid too much for our shares and could seek to immediately sell their shares to limit their losses."

Bargaining

The players bargain over a pie of size 1.

An <u>agreement</u> is a pair (x_1, x_2) where x_i is player i's share of the pie. The set of possible agreements is

$$X = \{(x_1, x_2) : x_1 + x_2 = 1\}$$

Player i prefers $(x_1, x_2) \in X$ to $(y_1, y_2) \in X$ if and only if $x_i > y_i$.

The bargaining protocol

The players can take actions only at times in the (infinite) set $T = \{0, 1, 2, ...\}$. In each $t \in T$ player i, proposes an agreement $x \in X$ and $j \neq i$ either accepts (Y) or rejects (N).

If x is accepted (Y) then the bargaining ends and x is implemented. If x is rejected (N) then the play passes to period $t + 1$ in which j proposes an agreement.

At all times players have perfect information. Every path in which all offers are rejected is denoted as disagreement (D). The only asymmetry is that player 1 is the first to make an offer.

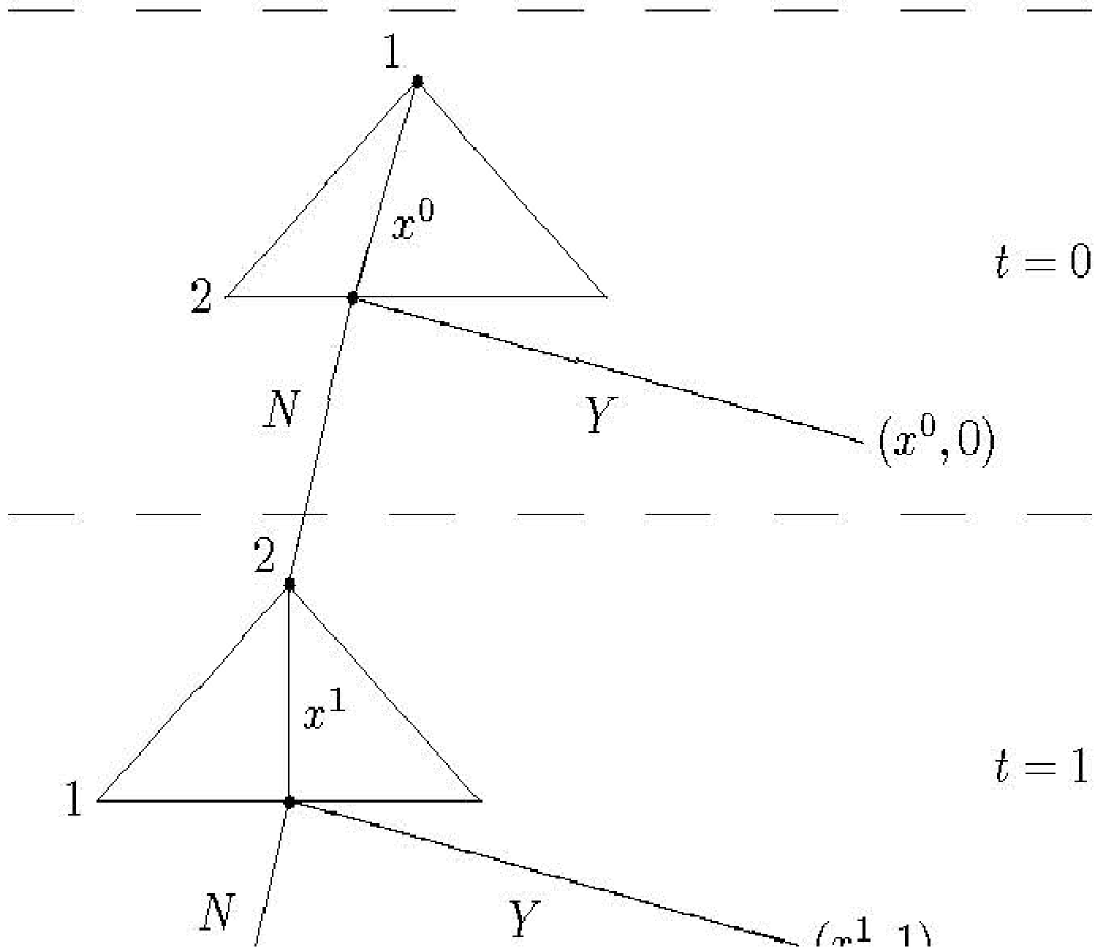
Preferences

Time preferences (toward agreements at different points in time) are the driving force of the model.

A bargaining game of alternating offers is

- an extensive game of perfect information with the structure given above, and

- player i's preference ordering $\succsim_i$over $(X \times T)$ are represented by $\delta_i^t u_i(x_i)$ for any $0 < \delta_i < 1$ where u_i is an increasing and concave function.


1
x^0
$t = 0$
2
N
Y
$(x^0, 0)$
2
x^1
$t = 1$
1
N
Y

Assumptions on preferences

A1 Disagreement is the worst outcome

For any $(x, t) \in X \times T$,

$$(x, t) \succsim_i D$$

for each i.

A2 Pie is desirable

– For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t) \text{ if and only if } x_i > y_i.$$

A3 <u>Time is valuable</u>

For any $t \in T$, $s \in T$ and $x \in X$

$$(x, t) \succsim_i (x, s) \text{ if } t < s$$

and with strict preferences if $x_i > 0$.

A4 <u>Preference ordering is continuous</u>

Let $\{(x_n, t)\}_{n=1}^{\infty}$ and $\{(y_n, s)\}_{n=1}^{\infty}$ be members of $X \times T$ for which

$$\lim_{n \to \infty} x_n = x \text{ and } \lim_{n \to \infty} y_n = y.$$

Then, $(x, t) \succsim_i (y, s)$ whenever $(x_n, t) \succsim_i (y_n, s)$ for all n.

A2-**A4** imply that for any outcome (x, t) either there is a unique $y \in X$ such that

$$(y, 0) \sim_i (x, t)$$

or

$$(y, 0) \succ_i (x, t)$$

for every $y \in X$.

A5 Stationarity

For any $t \in T$, $x \in X$ and $y \in X$

$$(x, t) \succ_i (y, t+1) \text{ if and only if } (x, 0) \succ_i (y, 1).$$

If $\succsim_i$satisfies **A2-A5** then player i's preference ordering $\succsim_i$over $(X \times T)$ are represented by

$$\delta_i^t u_i(x_i) \text{ where } 0 \leq \delta_i \leq 1$$

Present value

For $i = 1, 2$ we call $v_i(x_i, t)$ player i's <u>present value</u> of (x, t)

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y, 0) \sim_i (x, t) \\ 0 & \text{if } (y, 0) \succ_i (x, t) \text{ for all } y \in X. \end{cases}$$

Note that

$$(y, t) \succ_i (x, s) \text{ whenever } v_i(y_i, t) > v_i(x_i, s).$$

Delay

A6 Increasing loss to delay

$x_i - v_i(x_i, 1)$ is an increasing function of x_i.

If $\succsim_i$for each i satisfies **A2-A6**, then there exist a unique pair $(x^*, y^*) \in X \times X$ such that

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

Examples

[1] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = \delta_i^t x_i$$

where $\delta_i \in (0, 1)$.

[2] For every $(x, t) \in X \times T$

$$U_i(x_i, t) = x_i - c_i t$$

where $c_i > 0$ (constant cost of delay).

Although **A6** is violated, when $c_1 \neq c_2$ there is a unique pair $(x, y) \in X \times X$ such that $y_1 = v_1(x_1, 1)$ and $x_2 = v_2(y_2, 1)$.

Subgame perfect equilibrium

Any bargaining game of alternating offers in which players' preferences satisfy **A1**-**A6** has a unique SPE which is the solution of the following equations

$$y_1^* = v_1(x_1^*, 1) \text{ and } x_2^* = v_2(y_2^*, 1).$$

Note that if $y_1^* > 0$ and $x_2^* > 0$ then

$$(y_1^*, 0) \sim_1 (x_1^*, 1) \text{ and } (x_2^*, 0) \sim_2 (y_2^*, 1).$$

The equilibrium strategy profile is given by

Player 1	proposes	x^*
	accepts	$y_1 \geq y_1^*$
Player 2	proposes	y^*
	accepts	$x_1 \leq x_1^*$

The unique outcome is that player 1 proposes x^* in period 0 and player 2 accepts.

The structure of the model is asymmetric only in one respect: player 1 is the first to make an offer.

Recall that with constant discount rates the equilibrium condition implies that

$$y_1^* = \delta_1 x_1^* \text{ and } x_2^* = \delta_2 y_2^*$$

so that

$$x^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right) \text{ and } y^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right).$$

Thus, if $\delta_1 = \delta_2 = \delta$ $(v_1 = v_2)$ then

$$x^* = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right) \text{ and } y^* = \left(\frac{\delta}{1+\delta}, \frac{1}{1+\delta}\right)$$

so player 1 obtains more than half of the pie.

By shrinking the length of a period by considering a sequence of games indexed by Δ in which $u_i = \delta_i^{\Delta t} x_i$ we have

$$\lim_{\Delta \to 0} x^*(\Delta) = \lim_{\Delta \to 0} y^*(\Delta) = \left(\frac{\log \delta_2}{\log \delta_1 + \log \delta_2}, \frac{\log \delta_1}{\log \delta_1 + \log \delta_2}\right)$$

(l'Hôpital's rule).

Nash bargaining
(the axiomatic approach)

Bargaining

Nash's (1950) work is the starting point for formal bargaining theory.

The bargaining problem consists of

- a set of utility pairs that can be derived from possible agreements, and
- a pair of utilities which is designated to be a disagreement point.

Bargaining solution

The bargaining solution is a function that assigns a unique outcome to every bargaining problem.

Nash's bargaining solution is the first solution that

– satisfies four plausible conditions, and

– has a simple functional form, which make it convenient to apply.

A bargaining situation

A bargaining situation:

- N is a set of players or bargainers,
- A is a set of agreements/outcomes,
- D is a disagreement outcome, and

$\langle S, d \rangle$ is the primitive of Nash's bargaining problem where

- $S = (u_1(a), u_2(a))$ for $a \in A$ the set of all utility pairs, and $d = (u_1(D), u_2(D))$.

A <u>bargaining problem</u> is a pair $\langle S, d\rangle$ where $S \subset \mathbb{R}^2$ is compact and convex, $d \in S$ and there exists $s \in S$ such that $s_i > d_i$ for $i = 1, 2$. The set of all bargaining problems $\langle S, d\rangle$ is denoted by B.

A <u>bargaining solution</u> is a function $f : B \longrightarrow \mathbb{R}^2$ such that f assigns to each bargaining problem $\langle S, d\rangle \in B$ a unique element in S.

Nash's axioms

One states as axioms several properties that it would seem natural for the solution to have and then one discovers that the axioms actually determine the solution uniquely - Nash 1953 -

Does not capture the details of a specific bargaining problem (e.g. alternating or simultaneous offers).

Rather, the approach consists of the following four axioms: invariance to equivalent utility representations, symmetry, independence of irrelevant alternatives, and (weak) Pareto efficiency.

Invariance to equivalent utility representations (INV)

$\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i' \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ if

$$d_i' = \alpha_i d_i + \beta_i$$

and

$$S' = \{(\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2) \in \mathbb{R}^2 : (s_1, s_2) \in S\}.$$

Note that if $\alpha_i > 0$ for $i = 1, 2$ then $\langle S', d' \rangle$ is itself a bargaining problem.

If $\langle S', d' \rangle$ is obtained from $\langle S, d \rangle$ by the transformations

$$s_i \mapsto \alpha_i s_i + \beta_i$$

for $i = 1, 2$ where $\alpha_i > 0$ for each i, then

$$f_i(S', d') = \alpha_i f_i(S, d) + \beta_i$$

for $i = 1, 2$. Hence, $\langle S', d' \rangle$ and $\langle S, d \rangle$ represent the same situation.

Symmetry (SYM)

A bargaining problem $\langle S, d\rangle$ is symmetric if $d_1 = d_2$ and $(s_1, s_2) \in S$ if and only if $(s_2, s_1) \in S$. If the bargaining problem $\langle S, d\rangle$ is symmetric then

$$f_1(S, d) = f_2(S, d)$$

Nash does not describe differences between the players. All asymmetries (in the bargaining abilities) must be captured by $\langle S, d\rangle$.

Hence, if players are the same the bargaining solution must assign the same utility to each player.

Independence of irrelevant alternatives (IIA)

If $\langle S, d\rangle$ and $\langle T, d\rangle$ are bargaining problems with $S \subset T$ and $f(T, d) \in S$ then

$$f(S, d) = f(T, d)$$

If T is available and players agree on $s \in S \subset T$ then they agree on the same s if only S is available.

IIA excludes situations in which the fact that a certain agreement is available influences the outcome.

Weak Pareto efficiency (WPO)

If $\langle S, d \rangle$ is a bargaining problem where $s \in S$ and $t \in S$, and $t_i > s_i$ for $i = 1, 2$ then $f(S, d) \neq s$.

In words, players never agree on an outcome s when there is an outcome t in which both are better off.

Hence, players never disagree since by assumption there is an outcome s such that $s_i > d_i$ for each i.

$\underline{SYM \text{ and } WPO}$

restrict the solution on single bargaining problems.

$\underline{INV \text{ and } IIA}$

requires the solution to exhibit some consistency across bargaining problems.

Nash 1953: there is precisely one bargaining solution, denoted by $f^N(S, d)$, satisfying SYM, WPO, INV and IIA.

Nash's solution

The unique bargaining solution $f^N : B \to \mathbb{R}^2$ satisfying SYM, WPO, INV and IIA is given by

$$f^N(S, 0) = \underset{(s_1, s_2) \in S}{\arg\max} s_1 s_2$$

The solution is the utility pair that maximizes the product of the players' utilities.

Proof

Pick a compact and convex set $S \subset \mathbb{R}^2_+$ where $S \cap \mathbb{R}^2_{++} \neq \emptyset$.

Step 1: f^N is well defined.

– Existence: the set S is compact and the function $f = s_1 s_2$ is continuous.

– Uniqueness: f is strictly quasi-conacave on S and the set S is convex.

Step 2: f^N is the only solution that satisfies SYM, WPO, INV and IIA.

Suppose there is another solution f that satisfies SYM, WPO, INV and IIA.

Let

$$S' = \{(\frac{s_1}{f_1^N(S)}, \frac{s_2}{f_2^N(S)}) : (s_1, s_2) \in S\}$$

and note that $s'_1 s'_2 \leq 1$ for any $s' \in S'$, and thus $f^N(S', 0) = (1, 1)$.

Since S' is bounded we can construct a set T that is symmetric about the 45° line and contains S'

$$T = \{(a, b) : a + b \leq 2\}$$

By WPO and SYM we have $f(T, 0) = (1, 1)$, and by IIA we have $f(S', 0) = f(T, 0) = (1, 1)$.

By INV we have that $f(S', 0) = f^N(S', 0)$ if and only if $f(S, 0) = f^N(S, 0)$ which completes the proof.