Consumer Theory: The Mathematical Core

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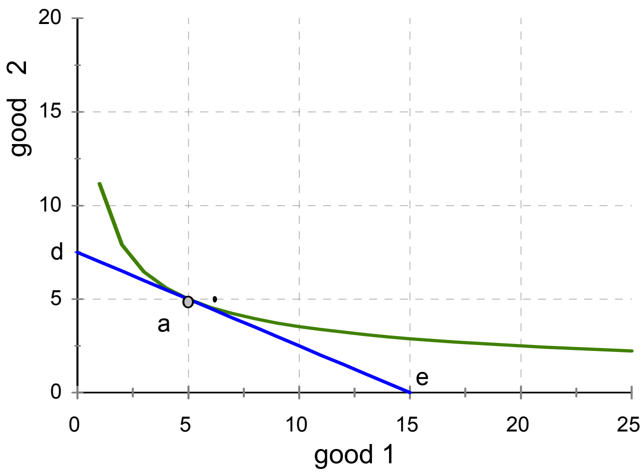
Suppose an individual has a utility function U(x) which is a function of non-negative commodity vectors x = $(x_1,x_2,...,x_N)$, and seeks to maximize this utility function subject to the budget constraint p·x = $p_1x_1 + p_2x_2 + ... + p_Nx_N \le y$, where y is income and p = $(p_1,p_2,...,p_N)$ is the vector of commodity prices. Define u = V(p,y) to be the value of utility attained by solving this problem; V is called the *indirect utility function*. Let x = X(p,y) = $(X^1(\text{p},y),X^2(\text{p},y),...,X^N(\text{p},y))$ be the commodity vector that achieves the utility maximum subject to this budget constraint. The function X(p,y) is called this consumer's *market* or *Marshallian* demand function. Since this function maximizes utility subject to the budget constraint, V(p,y) ≡ U(X(p,y)) ≡ $U(X^1(\text{p},y),X^2(\text{p},y),...,X^N(\text{p},y))$. The figure below shows for the case of two goods the budget line **d-e**, and the point **a** that maximizes utility.[1] Note that the point **a** is on the highest utility contour (or *indifference curve*) that touches the budget line, and that at **a** the indifference curve is tangent to the budget line, so that its slope, the *marginal rate of substitution* (MRS) is equal to the slope of the budget line, $-p_1/p_2$.



Note that the indirect utility function and the market demand functions all depend on income and on the price vector. For example, the demand for good 1, written out, is $x_1 = X^1(\text{p},y)$. When demand for good 1 is graphed against its own price p_1, then changes in own price p_1 correspond to movement *along* the graphed demand curve, while changes in income y or in the cross-price p_2 correspond to *shifts* in the graphed demand curve. If all income is spent, then p·X(p,y) ≡ y. The consumer is said to be *locally non-satiated* if in each neighborhood of a commodity vector there is always another that has strictly higher utility. When the consumer is locally non-satiated, no commodity vector that costs less than y can give the maximum obtainable utility level, the utility-maximizing consumer will spend all income,, and the indirect utility function V(p,y) is strictly increasing in y. Local non-satiation rules out "fat" indifference curves. A sufficient condition for local non-satiation is that the utility function be continuous and monotone increasing; i.e., if x′>> x″, then U(x′) > U(x″).

[1]The graph was created using the utility function U(x) = $x_1^{1/3}x_2^{2/3}$. The budget line **d-e** is given by $9x_1 + 18x_2 = 135$. The demand functions that maximize utility are $X^1(p_1,p_2,y) = y/3p_1$ and $X^2(p_1,p_2,y) = 2y/3p_2$. The indirect utility function is $V(p_1,p_2,y) = y(3p_1)^{-1/3}(3p_2/2)^{-2/3}$.

If income and all prices are scaled by a constant, the budget set and the maximum attainable utility remain the same. This implies that $V(p,y)$ and $X(p,y)$ are unchanged by rescaling of income and prices. Functions with this property are said to be *homogeneous of degree zero* in income and prices.

Given utility level u, let $y = M(p,u)$ be the minimum income needed to buy a commodity vector that gives utility u; M is called the *expenditure function*. Let $x = H(p,u) \equiv (H^1(p,u),H^2(p,u),...,H^N(p,u))$ be the commodity vector that achieves the minimum expenditure subject to the constraint that a utility level u be attained. Then,

$$M(p,u) \equiv p\cdot H(p,u) \equiv p_1H^1(p,u) + ... + p_NH^N(p,u).$$

The demand functions $H(p,u)$ are called the *Hicksian*, *constant utility*, or *compensated* demand functions. If all prices are rescaled by a constant, then the commodity vector solving the minimum income problem is unchanged, so that $H(p,u)$ is homogeneous of degree zero in p. The expenditure is scaled up by the same constant as the prices, and is said to be *linear homogeneous* in p. Suppose in the figure above, instead of fixing the budget line and locating the point **a** where the highest indifference curve touches this line, we had fixed the indifference curve and varied the level of income for given prices to find the minimum income necessary to touch this indifference curve. Provided the consumer is locally non-satiated so that all income is spent, this minimum again occurs at **a**. Then, solving the utility maximization problem for a fixed y and attaining a maximum utility level u, and solving the expenditure minimization problem for this u and attaining the income y, lead to the same point **a**. In this case, if the consumer utility level is $u = V(p,y)$, then y is the minimum income at which utility level u can be achieved. Then, the result that the utility maximization problem and the expenditure minimization problem pick out the same point **a** implies that $y \equiv M(p,V(p,y))$ and $u \equiv V(p,M(p,u))$; i.e., V and M are inverses of each other for fixed p. Further, $X(p,y) \equiv H(p,V(p,y))$ and $X(p,M(p,u)) \equiv H(p,u)$.

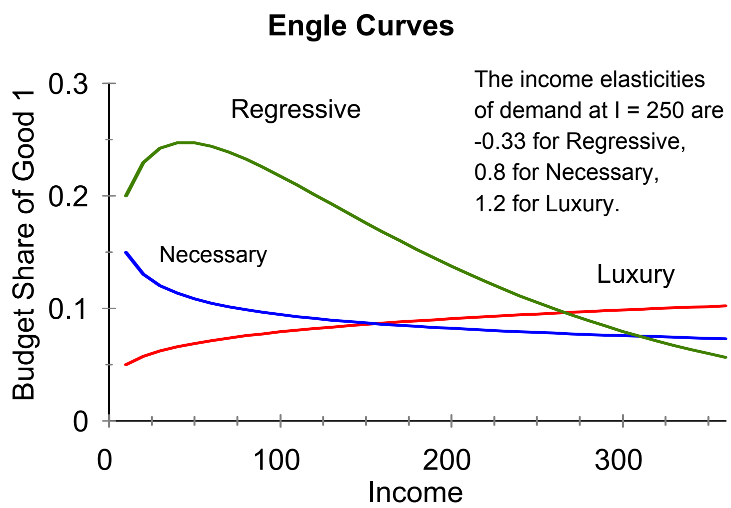
The demand for a commodity is said to be *normal* if demand does not fall when income rises, and *inferior* or *regressive* if demand falls when income rises. A commodity is a *luxury good* if its budget share rises when income rises, and is otherwise a *necessary good*. The budget share of the first commodity is $s_1 = p_1X^1(p,y)/y$. Define the *income elasticity of demand*,

$$\eta = (y/X^1(p,y))\cdot\partial X^1(p,y)/\partial y.$$

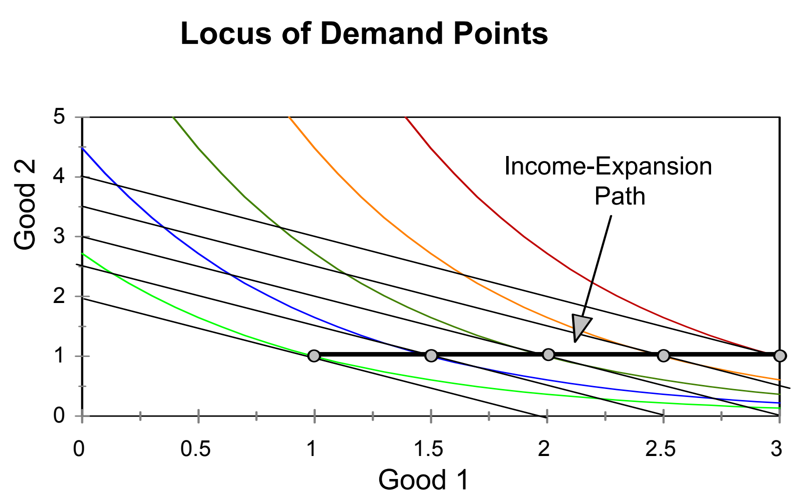
This is the percentage by which demand for good 1 increases when income goes up by one percent. Show as an exercise that the income elasticity of the budget share satisfies

$$(y/ s_1)\cdot\partial s_1/\partial y = \eta - 1.$$

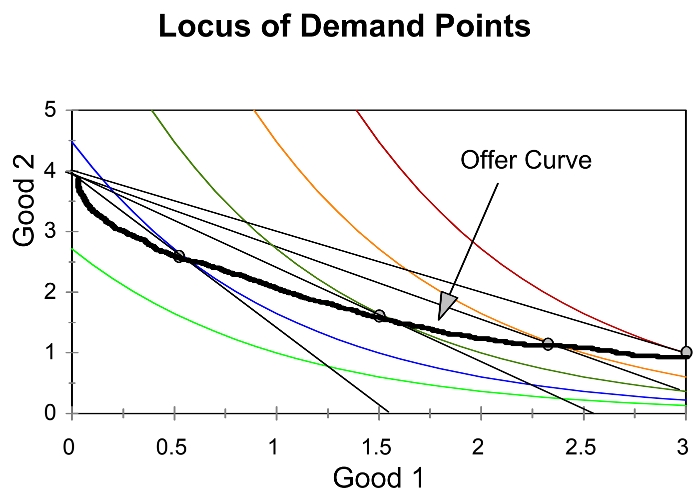
Then, a luxury good has an income elasticity of its budget share greater than zero, a necessary good has an income elasticity of the budget share less than zero, and an inferior good has an income elasticity of the budget share less than -1. A graph of the demand for a good against income is called an *Engle curve*. The figure below shows the Engle curves for three cases.



It is possible to trace out the locus of demand points in an indifference curve map as income changes with prices fixed; this locus is called an *income-offer curve* or *income-expansion path*. Points on an income-expansion path correspond to points on Engle curves for each of the commodities. The figure below shows an income-expansion path when good 1 is a luxury good.



In an indifference curve map, it is also possible to trace out the locus of demand points as the price of a good changes; this locus is called an *price offer curve*. The figure below depicts a typical price offer curve. Note that the price-offer curve is the locus of tangencies between indifference curves and budget lines that pivot about one point on the vertical axis, in this case (0,4). As one moves out along the offer curve, one is identifying quantities demanded of good 1 as its price falls. In this diagram, falling price always results in increased demand for good 1. This is called the *"law of demand"*, but a qualification is needed to make this law hold, as described below.

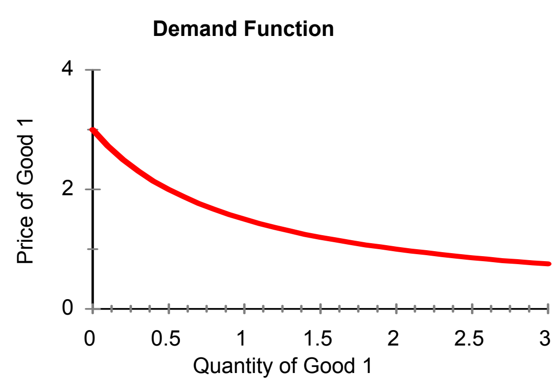


If one plots the demand for good 1 against its own price, one obtains the standard picture of a demand function, given in the next figure. Remember that this is the locus of points satisfying $x_1 = X^1(\mathrm{p},y)$ as p_1 changes, with y and $p_2,\ldots,p_N$ kept fixed. If y or $p_2,\ldots,p_N$ were to change, this would appear as a shift in the demand function given in the picture.

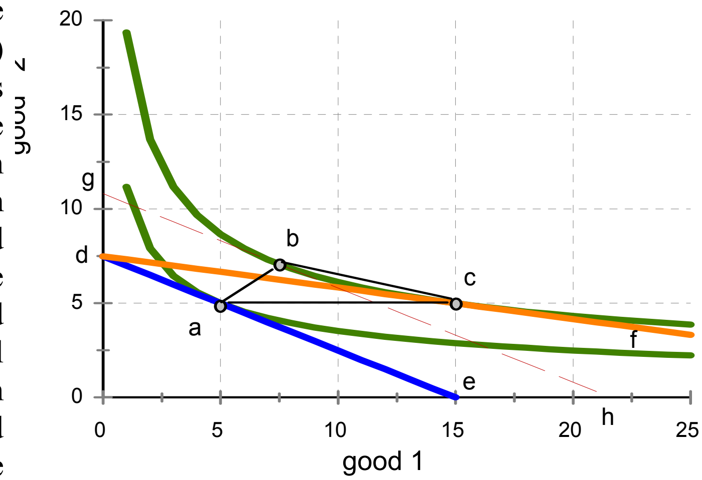
It is often useful to characterize demand functions in terms of their *own-price elasticity*, defined as

$$\epsilon = (p_1/X^1(\mathrm{p}))\cdot\partial X^1(\mathrm{p})/\partial p_1.$$

In this definition, ϵ is negative in the usual case that the demand function slopes downward. When $\epsilon < -1$, demand is said to be *elastic*, and when $-1 < \epsilon < 0$, demand is said to be *inelastic*. Verify as an exercise that the own-price elasticity of the budget share of good 1 is $1 + \epsilon$.



If one looks in more detail in an indifference curve map at the change in demand for a good resulting from a decrease in its price, one can decompose the total change into the effects of a pure income change, with relative prices fixed, and a pure price change, with income compensated to keep utility constant. The first of these effects is termed an *income effect*, and the second is termed a *substitution effect*. What we will show is that the substitution effect always operates to increase the demand for a good whose price falls. We will show that for a normal good, the income effect reinforces the substitution effect, so that falling price increases demand. However, for an inferior good, we will show that the substitution and income effects work in opposite directions. It is even possible for the income effect to be larger in magnitude than the income effect, so that a fall in price reduces demand. In these circumstances, the good is called a *Giffen good*. The occurrence of Giffen goods is primarily a curiosity rather than a common event, and requires that a good have a large share of the budget and be strongly inferior. An example of a Giffen good is day-old bread, which may absorb most of the income of a homeless person. If the price of day-old bread falls, this consumer needs to commit less income to subsistence on day-old bread, and may instead buy more fresh bread. The figure below shows the decomposition of a total price effect into income and substitution effects. In this figure, the good is normal, so that the income and substitution effects are reinforcing. In the diagram, start from the budget line **d-e**, for which utility is maximized at **a**. Now decrease the price of good 1 so the budget line becomes **d-f**. On this new budget line, utility is maximized at **c**. The total price effect is **a-c**. In this case, the total own price effect is to increase demand for good 1 (from **a** to **c**) when the price of good 1 falls. Now introduce the budget **g-h** which keeps prices the same as the were at **d-e**, but compensates income so that maximum utility (attained at **b**) is the same as it is at **c**. Then, **a-b** is the income effect, and **b-c** is the substitution effect. The substitution effect always operates in the direction of increasing demand for the good whose price is decreasing. The income effect also increases demand for good 1 in this case, since it is a normal good. However, if good 1 had been an inferior good, the income effect would partially (or completely) offset the substitution effect.



The statement we obtained geometrically that the substitution effect always increases the demand for a good whose price is falling can also be established mathematically, with great generality. The remainder of these notes provide a simple mathematical exposition of this analysis.

From the definition of the expenditure function, for any vector q, and ε a small scalar,

$$M(p+\varepsilon q,u) \equiv (p+\varepsilon q)\cdot H(p+\varepsilon q,u) \le (p+\varepsilon q)\cdot H(p,u) = M(p,u) + \varepsilon q\cdot H(p,u),$$

or

$$M(p+\varepsilon q,u) - M(p,u) \le \varepsilon q\cdot H(p,u).$$

This implies

$$\lim_{\varepsilon\searrow 0}(M(p+\varepsilon q,u) - M(p,u))/\varepsilon \le q\cdot H(p,u) \le \lim_{\varepsilon\nearrow 0}(M(p+\varepsilon q,u) - M(p,u))/\varepsilon.$$

But the left and right hand ends of this inequality both converge to $q\cdot\nabla_p M(p,u)$ when the derivative exists, establishing that $q\cdot H(p,u) = q\cdot\nabla_p M(p,u)$. This is true for any q, so $H(p,u) \equiv \nabla_p M(p,u)$. This is called *Shephard's identity*. From this, $X(p,y) \equiv H(p,V(p,y)) \equiv \nabla_p M(p,u)|_{u=V(p,y)}$.

From the definition of the expenditure function,

$$M(p+q.u) = (p+q)\cdot H(p+q.u) = p\cdot H(p+q,u) + q\cdot H(p+q,u) \ge M(p,u) + M(q,u).$$

A linear homogeneous function that satisfies this inequality is said to be *concave*. Mathematical implications of concavity are that M will be continuous and will almost always have first and second derivatives, the mixed second partial derivatives will be independent of the order of differentiation, and the own second partial derivatives will be non-positive. As a consequence, the Hicksian demand function exists and satisfies Shephard's identity almost everywhere in p, without any requirement that the utility function be quasi-concave.

Consider budgets(p',y') and (p'',y''), and the convex combination $p^* = \theta p'+(1-\theta)p''$ and $y^* = \theta y'+(1-\theta)y''$ with $0 < \theta < 1$. We now demonstrate that $V(p^*,y^*) \le \max\{V(p',y'),V(p'',y'')\}$. A function with this property is said to be *quasi-convex*. Suppose x^* maximizes utility subject to the constraint $p^*\cdot x \le y^*$, so that $V(p^*,y^*) = U(x^*)$. Then $\theta(p'\cdot x^*-y') +(1-\theta)p''\cdot x^*-y'') \le 0$, implying that either $p'\cdot x^* \le y'$ and hence $U(x^*) \le V(p',y')$, or else $p''\cdot x^* \le y''$ and hence $U(x^*) \le V(p'',y'')$. Therefore, $V(p^*,y^*) = U(x^*) \le \max\{V(p',y'),V(p'',y'')\}$, establishing the result.

From the identity $y \equiv M(p,V(p,y))$ and the result $H(p,u) = \nabla_p M(p,u)$, one can take derivatives with respect to y and p to obtain

$$1 \equiv \nabla_u M(p,u))|_{u=V(p,y)}\ V_y(p,y) \text{ and } 0 \equiv \nabla_u M(p,u)|_{u=V(p,y)}\cdot\nabla_p V(p,y) + \nabla_p M(p,u)|_{u=V(p,y)}.$$

Then, using the first of these equations to eliminate the term $\nabla_u M(p,u)|_{u=V(p,y)}$ in the second gives a relationship between the indirect utility function and the Marshallian demand functions that is termed *Roy's identity*,

$$X^1(p,y) \equiv H^1(p,V(p,y)) \equiv \partial M(p,u)/\partial p_1|_{u=V(p,y)} \equiv -(\partial V(p,y)/\partial p_1)/(\partial V(p,y)/\partial y).$$

From the identity $X^1(p,y) \equiv H^1(p,V(p,y))$, by taking derivatives one obtains the relationships $\nabla_y X^1(p,y) \equiv \nabla_u H^1(p,u)|_{u=V(p,y)}\nabla_y V(p,y)$ and $\nabla_p X^1(p,y) \equiv \nabla_u H^1(p,u)|_{u=V(p,y)}\nabla_p V(p,y) + \nabla_p H^1(p,u)|_{u=V(p,y)}$. Use the first equation to obtain $\nabla_u H^1 = \nabla_y X^1/\nabla_y V$, and then substitute this into the second equation to obtain

$$\begin{aligned}\nabla_p X^1(p,y) &\equiv [\nabla_p V(p,y)/\nabla_y V(p,y)]\nabla_y X^1(p,y) + \nabla_p H^1(p,u)|_{u=V(p,y)} \\ &\equiv - X(p,y)\nabla_I X^1(p,y) + \nabla_p H^1(p,u)|_{u=V(p,y)}.\end{aligned}$$

Written out, this equation decomposes the overall effect of own price on market demand into a *substitution effect* $\partial H^1(p,u)/\partial p_1|_{u=V(p,y)} = \partial^2 M(p,u)/\partial p_1^{\,2}|_{u=V(p,y)}$, which corresponds to the change in demand when income is compensated to keep the consumer on the same indifference curve, and is always non-positive, and an *income effect*, which for the own price change is $- X^1(p,y)\nabla_y X^1(p,y)$. If commodity 1 is normal, its change with income, $\nabla_y X^1(p,y)$, is non-negative, and the income effect reinforces the non-positive substitution effect so that demand for commodity 1 falls when its price rises, and the "law of demand" holds. If commodity 1 is inferior, its change with income is negative, and the income effect tends to offset the non-positive substitution effect. Usually, the income effect for an inferior good is not strong enough to completely offset the substitution effect.

An interesting variant of the classical consumer demand problem occurs when instead of considering fixed income y when prices change, one considers income $y = \Pi(p,t)$ that is a function of prices and of policies (denoted by t) that control taxes and transfers which can alter income. For example, one might consider a consumer who has an initial endowment of goods ω, and receives income $\Pi(p,t) = p\cdot\omega$ from selling this endowment in the market. Endowments of leisure that are sold as labor are a leading case. If one decomposes the effect on demand of a change in price into substitution and income effects, one gets as before a substitution effect that never increases demand for a good when its price rises, but now the income effect has the pattern given in the following table:

Income Effect	Consumer Net Demander of Good	Consumer Net Supplier of Good
Normal Good	Reinforces substitution effect	Offsets substitution effect
Inferior Good	Offsets substitution effect	Reinforces substitution effect