Problem Set #4

ECONOMICS 241A
SPRING 2006

Due May 3

1. A least-squares based alternative to quantile regression is the "asymmetric least squares" (ALS) estimator $\hat{\beta}_{ALS}$, which is defined to minimize

$$S(\beta) \equiv \frac{1}{n}\sum_{i=1}^{n}(y_i - x_i'\beta)^2 \cdot \left|\pi - 1\{y_i < x_i'\beta\}\right|$$

over β, where π is some pre-specified number between zero and one, and y_i and x_i are jointly i.i.d. and assumed to satisfy the linear model

$$y_i = x_i'\beta_0 + \varepsilon_i, \qquad i = 1, ..., n,$$

with ε_i assumed to be independent of x_i. A special case of this estimator is the classical least squares estimator (with $\pi = 1/2$).

Assuming there is an intercept term in β_0, which is normalized so that

$$0 = \arg\min_b E[(\varepsilon_i - b)^2 \cdot |\pi - 1\{\varepsilon_i < b\}|],$$

and assuming that all needed moments of y_i and x_i are finite, discuss any "identification" conditions needed for consistency of $\hat{\beta}_{ALS}$ for β_0, and derive the asymptotic distribution of $\hat{\beta}_{ALS}$.

2. Suppose y_i satisfies a Box-Cox transformation model with right-censoring: that is,

$$y_i = min\{h(\mathbf{x}_i'\beta_0 + \varepsilon_i; \lambda_0), c\},$$

where the constant c is known and the "inverse Box-Cox tranform" $h(\cdot)$ is defined as

$$h(y;\lambda) \equiv (1+\lambda y)^{1/\lambda} \qquad \text{if } \lambda \neq 0, \qquad h(y;0) \equiv exp\{y\}.$$

A. Assume ε_i is independent of $\mathbf{x}_i$ with c.d.f. and density functions $F(\cdot;\tau_0)$ and $f(\cdot;\tau_0)$, respectively; also, assume that $\Pr\{\mathbf{x}_i'\beta_0 + \varepsilon_i > 0\} = 1$ for all possible values of β_0 (so that y_i is well-defined with probability one). Under these assumptions, derive the average log-likelihood for an i.i.d. sample of N observations on y_i and $\mathbf{x}_i$.

B. Now suppose that there is no right-censoring (i.e., $c = \infty$), but that the parametric form of the density for ε_i is unknown, and that ε_i is known only to satisfy the conditional moment restriction

$$E[\varepsilon_i | x_i] = 0$$

(so that ε_i is no longer required to be independent of x_i). What is the smallest asymptotic covariance matrix of a regular $\sqrt{n}$-consistent estimator of β_0 and λ_0 that uses only this restriction? What is the form of the "optimal instrument" vector for this conditional moment restriction?

C. Assuming now that $c < \infty$ and that ε_i is continuously distributed with conditional median zero,

$$E[sgn\{\varepsilon_i\}|x_i] = 0,$$

propose a $\sqrt{n}$-consistent estimator of the parameters that only exploits this restriction, and derive the form of its asymptotic distribution.

3. Consider a "middle censoring" model in which a linear latent dependent variable

$$y_i^* = x_i'\beta + \varepsilon_i$$

is related to an observable dependent variable y_i by the following transformation:

$$y_i = 1\{y_i^* < a\}(y_i^* - a) + 1\{b < y_i^*\}(y_i^* - b),$$

where a and b are known constants with $a < b$. Thus, $y_i = 0$ when y_i^* is in the inteval $[a, b]$, and y_i is "uncensored" when y_i^* is outside that interval.

A. Assuming the error term ε_i is normally distributed, $\varepsilon_i \sim N(0, \sigma^2)$, and is independent of the regressor vector x_i, derive the average log likelihood function $L_n(\beta, \sigma^2)$ for a sample of n i.i.d. observations on y_i and x_i.

B. Describe an alternative consistent estimator to maximum likelihood for the β parameters in this structural equation. You do not need to show consistency or asymptotic normality, but you should precisely state any assumptions on ε_i that are required for consistency, and give an explicit form of any functions to be maximimized or minimized or any equations to be solved in computing your proposed estimator.