Final Midterm Exam

ECONOMICS 241A
SPRING 2006

May 9, 2006

Instructions: This is a 20 point exam, with equal weights for each question. The answers must be turned in no later than 25 hours after you pick up the exams, to Jim Powell (669 Evans). You may consult and cite any lecture notes and any of the references on the syllabus; you may not cite any other outside source, and under no circumstances should you discuss the exam with anyone other than the instructor before you submit your answers. Please make your answers elegant – that is, clear, concise, and, above all, correct.

Suppose a sample of N i.i.d. observations on a scalar dependent variable y_i and p-dimensional vector of (non-constant) regressors x_i satifies a linear model

$$y_i = x_i'\beta_0 + \varepsilon_i,$$

where the slope coefficients β_0 are unknown, and the unobservable error term ε_i is statistically independent of the regressors x_i, with (unknown) marginal density function $f(\varepsilon)$ that is very well behaved (i.e., having lots of continuous derivatives, with the level and derivatives of f being uniformly bounded).

A "rank regression" estimator of the slope coefficient vector β_0 is defined to minimize the sum of absolute deviations of differences in dependent variables $y_i - y_j$ and corresponding differences in regression functions $(x_i - x_j)'\beta$ across all distinct pairs of observations; that is,

$$\begin{aligned}
\hat{\beta} &\equiv \arg\min_{\beta \in R^p} S_n(\beta), \\
S_n(\beta) &\equiv \binom{N}{2}^{-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \left|(y_i - y_j) - (x_i - x_j)'\beta\right|.
\end{aligned}$$

1. Give an argument for consistency of $\hat{\beta}$ for β_0 under the assumptions on the model given above, using analogous arguments to those for consistency of the LAD estimator of regression coefficients under a conditional median restriction.

2. The approximate first-order condition for the minimization problem defining $\hat{\beta}$ is

$$\begin{aligned}
\hat{\Psi}_N(\hat{\beta}) &\equiv \binom{N}{2}^{-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} sgn\left\{(y_i - y_j) - (x_i - x_j)'\hat{\beta}\right\} \cdot (x_i - x_j) \\
&\equiv \binom{N}{2}^{-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} sgn\left\{\hat{\varepsilon}_i - \hat{\varepsilon}_j\right\} \cdot (x_i - x_j) \\
&= o_p\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}$$

where, as usual,

$$sgn\{u\} \equiv 1\{u \geq 0\} - 1\{u \leq 0\}$$

and

$$\hat{\varepsilon}_i \equiv y_i - x_i'\hat{\beta}.$$

Show that this condition is equivalent to a sample moment condition which sets the sample covariance of the regressors and the ranks of the residuals to zero (approximately); that is, show that

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\hat{R}_i}{N+1} - \frac{1}{2}\right) \cdot x_i = o_p\left(\frac{1}{\sqrt{N}}\right),$$

where

$$\hat{R}_i \equiv \sum_{j=1}^{N} 1\{\hat{\varepsilon}_j \leq \hat{\varepsilon}_i\}$$

is the rank of the i^{th} residual in the sample, with

$$\sum_{\iota=1}^{N} \hat{R}_i = \left(\begin{array}{c} N+1 \\ 2 \end{array}\right) = \frac{N(N+1)}{2}$$

(ignoring possible ties in the residuals). [You should convert the U-statistic $\hat{\Psi}_N$ into the corresponding V-statistic before starting on the algebra.]

3. Rewriting the U-process $\hat{\Psi}_N(\beta)$ characterizing the first-order condition as

$$\hat{\Psi}_N(\beta) \equiv \left(\begin{array}{c} N \\ 2 \end{array}\right)^{-1} \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \rho(z_i, z_j; \beta)$$

(with $z_i \equiv (y_i, x_i')'$), assume (without proof) that the estimator $\hat{\beta}$ also solves the approximate moment condition

$$\tilde{\Psi}_N(\hat{\beta}) = o_p\left(\frac{1}{\sqrt{N}}\right),$$

where $\tilde{\Psi}_N(\beta)$ is the projection of the U-process,

$$\begin{aligned} \tilde{\Psi}_N(\beta) &\equiv \frac{1}{N}\sum_{i=1}^{N} \psi(z_i; \beta), \\ \psi(z_i; \beta) &\equiv E\left[\rho(z_i, z_j; \beta) | z_i\right]. \end{aligned}$$

Calculate the form of the function ψ, and use this expression to derive the limiting normal distribution of $\sqrt{N}(\hat{\beta} - \beta_0)$ under the given assumptions. Your expression for the asymptotic covariance matrix should involve the nuisance parameter

$$\tau_0 \equiv E[f(\varepsilon_i)] = \int [f(u)]^2 du.$$

[Hint: you will need to use the fact that, if $F(u)$ is the c.d.f. of ε_i, then $F(\varepsilon_i) \equiv u_i$ is uniformly distributed on $(0, 1)$.]

4. If $\hat{f}(\varepsilon)$ is the kernel density estimator of $f(\varepsilon)$ using the residuals $\hat{\varepsilon}_i$, i.e.,

$$\hat{f}(\varepsilon) = \frac{1}{Nh_N} \sum_{j=1}^{N} K\left(\frac{\varepsilon - \hat{\varepsilon}_j}{h_N}\right),$$

with K a smooth, symmetric, nonnegative kernel with bounded derivatives, an estimator of the nuisance parameter τ_0 is

$$\hat{\tau} \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{f}(\hat{\varepsilon}_i).$$

Show that if $h_N \to 0$ and $h_N^2 \cdot \sqrt{N} \to \infty$ as $N \to \infty$, then $\hat{\tau}$ is (weakly) consistent for τ_0. [First use a mean-value expansion to show that the residuals $\hat{\varepsilon}_i$ can be replaced by the true error terms ε_i, then use analogous bias-variance calculations to those for the kernel density estimator $\hat{f}(\varepsilon)$ itself.]