Final Midterm Exam

ECONOMICS 241A

Spring 2008

Instructions: This is a 40 point exam, with given weights for each question; all subsections have equal weight. The answers must be turned in no later than 25 hours after you pick up the exams, to Jim Powell (669 Evans) or Carol Smallfield (673 Evans). You may consult and cite any lecture notes and any of the references on the syllabus; you may not cite any other outside source, and under no circumstances should you discuss the exam with anyone other than the instructor before you submit your answers. Please make your answers elegant – that is, clear, concise, and, above all, correct.

1. (15 points) For a linear equation $y_i = \beta \cdot x_i + \varepsilon_i$ with i.i.d. observations on y_i and x_i (both scalar), suppose a certain economic theory yields the conditional moment restriction that the second moment of the error terms ε_i equals one, i.e.

$$E[\varepsilon_i^2 | x_i] \equiv 1, \qquad a.s.$$

   (a) What would the form of the unconditional moment restriction for the optimal (infeasible) GMM estimator of β be for this restriction, assuming that the parameter space B is small enough that β_0 is identified using just this restriction? Under what condition on the distribution of ε_i given x_i would the corresponding estimator be root-n consistent and asymptotically normal, and what is its asymptotic distribution be in this case? To obtain a feasible version of the optimal estimator, which functions might need to be estimated nonparametrically?

   (b) Show that, if the parameter space B is the whole real line, the parameter β is not generally identified using just the optimal unconditional moment restriction derived above for some joint distributions of the data.

   (c) Suppose that, to address the identification problem when $B = R$, you are willing to impose the additional conditional moment restriction

$$E[\varepsilon_i | x_i] = 0.$$

   With both this extra "identifying" restriction and with the original restriction $E[\varepsilon_i^2 | x_i] = 1$, derive the form of the optimal IV estimator for these conditional moment restrictions, and find the asymptotic covariance matrix of the corresponding estimator.

2. (25 points) For a sample of N i.i.d. observations on a scalar dependent variable y_i and p-dimensional vector of (non-constant) regressors x_i, the conditional median of y_i given $x_i = x_0$, denoted $m_0 \equiv m(x_0)$, is any value that satifies

$$\begin{aligned} \Pr\{y_i &\leq m_0 \mid x_i = x_0\} \geq 1/2, \\ \Pr\{y_i &\geq m_0 \mid x_i = x_0\} \geq 1/2. \end{aligned}$$

Assume the random vector $z_i \equiv (y_i, x_i')'$ is jointly continuously distributed, with conditional density $\phi(y|x)$ of y_i given $x_i = x$ and marginal density $f(x)$ of x_i which are both smooth and well-behaved (e.g, permit interchange of limits and expectations and the usual series expansions for nonparametric estimation).

(a) Give conditions on the densities above which ensure that the parameter $m_0 \equiv m(x)$ is uniquely determined as the solution to the conditional extremum problem

$$m_0 \equiv \arg\min_{b \in R} E[(|y_i - b| - |y_i|) \mid x_i = x_0].$$

(b) A kernel estimator of m_0 can be defined to minimize a kernel-weighed average of absolute deviations of differences $y_i - b$ over b; that is,

$$\begin{aligned} \hat{m} &\equiv \arg\min_{b \in R} S_n(b), \\ S_n(b) &\equiv \frac{1}{Nh^p} \sum_{i=1}^{N} K\left(\frac{x_0 - x_i}{h}\right) \cdot |y_i - b|, \end{aligned}$$

where the kernel function $K(\cdot)$ satisfies standard regularity conditions – it integrates to one, is nonnegative, symmetric about zero, bounded, and smooth – and the nonrandom bandwidth sequence $h = h_N$ satisfies $h \to 0$, $Nh^p \to \infty$, $Nh^{p+4} \to 0$ as $N \to \infty$.Under these conditions, and the conditions used in part (a) above, give an argument for consistency of $\hat{m}$ for $m \equiv m(x)$ using analogous arguments based on those for consistency of the LAD estimator of an unconditional median and for consistency of kernel estimators of density and regression functions.

(c) Suppose you have established the following approximate first-order condition for the minimization problem defining $\hat{m}$:

$$\begin{aligned} \hat{\Psi}_N(\hat{m}) &\equiv \frac{1}{Nh^p} \sum_{i=1}^{N} K\left(\frac{x_0 - x_i}{h}\right) \cdot sgn\left\{y_i - \hat{m}\right\} \\ &= o_p\left(\frac{1}{\sqrt{Nh^p}}\right), \end{aligned}$$

where

$$sgn\{u\} \equiv 1\{u \geq 0\} - 1\{u \leq 0\}.$$

Suppose you have also established a "stochastic equicontinuity" result that

$$\hat{\Psi}_N(\hat{m}) - \hat{\Psi}_N(m_0) - [\Lambda(\hat{m}) - \Lambda(m_0)] = o_p\left(\frac{1}{\sqrt{Nh^p}}\right),$$

where

$$\Lambda(m) = \lim_{N \to \infty} E[\hat{\Psi}_N(m)].$$

Use these results to derive an explicit form for the asymptotic (normal) distribution of $\hat{m}$. You need not verify the conditions of the limit theorems you use; instead, use existing results on the asymptotics of kernel density and regression estimators wherever possible.

(d) Give an algebraic form for an asymptotic confidence interval for m_0 of the form

$$CI \equiv [\hat{m} - SE(\hat{m}), \hat{m} + SE(\hat{m})]$$

which satisfies

$$\Pr\{m_0 \in CI\} \to 95\%$$

as $N \to \infty$. Give an explicit form for $SE(\hat{m})$ as a function of the sample size and consistent estimators of any nuisance parameters. (Again, you need not verify consistency explicitly, but should use existing results from the nonparametric literature.)

(e) Without going into details, briefly state how the form of the estimator of m_0 and its asymptotic distribution would change if m_0 were the π^{th} quantile of y_i given $x_i = x_0$, i.e.,

$$\begin{aligned} \Pr\{y_i &\leq m_0 \mid x_i = x_0\} \geq \pi, \\ \Pr\{y_i &\geq m_0 \mid x_i = x_0\} \geq (1-\pi), \end{aligned}$$

for some $\pi \in (0,1)$.