Problem Set 3

Due: Tuesday, October 9, 2007

1. Consider the time-series model

$$y_t = \alpha y_{t-1} + \beta x_t + \varepsilon_t + \gamma \varepsilon_{t-1}$$

where the ε's are normal mean-zero white noise, the x's are exogenous, and $|\alpha|$ and $|\gamma|$ are less than one. Suppose the model is estimated from a sample of size T using the Gauss-Newton method for maximizing the approximate likelihood. Find an expression for the score test statistic of the hypothesis that $\gamma = 0$. How does it differ from T times the squared first-order autocorrelation coefficient of the OLS residuals.?

2. Consider the ARMA(1,1) process

$$y_t = \alpha y_{t-1} + \varepsilon_t + \beta \varepsilon_{t-1}$$

where the ε's are i.i.d. N(0,1) random variables. Assume $|\alpha|$ and $|\beta|$ are less than one. You observe $y_1, ..., y_T$.

   (a) Consider the family of instrumental variable estimators

$$a_p = \frac{\sum_t y_{t-p} y_t}{\sum_t y_{t-p} y_{t-1}},$$

   where p is a positive integer and the sum is over the last $T - p$ observations. Assuming without proof that sample moments converge to population moments, show that the OLS estimator is consistent for α only if $\beta = 0$. Show that, when $p \geq 2$, a_p is consistent as long as $\alpha \neq 0$ and $\alpha \neq -\beta$.

   (b) Assuming that $T^{-1/2} \sum_t y_{t-p} \varepsilon_t$ is asymptotically normal, find the variance of the limiting distribution of $\sqrt{T}(a_p - \alpha)$ when $p \geq 2$. [Hint: $T^{-1/2} \sum_t y_{t-p}(\varepsilon_t + \beta \varepsilon_{t-1}) \approx T^{-1/2} \sum_t (y_{t-p} + \beta y_{t-p+1}) \varepsilon_t$.]

   (c) Show that the variance is minimized when $p = 2$.

3. Symmetric moving averages are often used to eliminate "noise" in a data series. For example, the series $\{x_t\}$ might be replaced by the filtered version $\{y_t\}$ where

$$y_t = \frac{1}{2p+1} \sum_{j=-p}^{p} x_{t-p}$$

and p is some nonnegative integer. An alternative is: apply the formal Fourier transform to $\{x_t\}$ producing $\tilde{x}(\lambda) = \frac{1}{2\pi} \sum_t x_t e^{-i\lambda t}$; next, set to zero all the values of $\tilde{x}(\lambda)$ corresponding to frequencies outside the range $(-b, b)$ for some value b in the interval $(0, \pi)$; finally, apply the inverse Fourier transform to get a new series $\{z_t\}$. Assuming convergence of the Fourier transform, the new series will be a linear filter of x_t. That is, $z_t = \sum c_j x_{t-j}$ because $\tilde{z}(\lambda) = b(\lambda)\tilde{x}(\lambda)$ and Fourier transforms convert convolutions into products. (In fact, by using the Stieltjes integral version of the transform, this procedure can be justified even if the formal Fourier series does not converge.)

   (a) What is the function $b(\lambda)$?

   (b) Find the c_j using the inversion rule $c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(\lambda) e^{i\lambda j} d\lambda$.

4. Consider the harmonic process $y_t = A\cos(\theta t) + B\sin(\theta t)$, where θ is a constant in $(0, \pi)$; A and B are uncorrelated zero-mean random variables with variance one.

   (a) Using the fact that $\cos(X) = (e^{ix}+e^{-ix})/2$ and $\sin(X) = (e^{ix}-e^{-ix})/2i$, show that $\cos(X-Z) = \cos(X)\cos(Z) + \sin(X)\sin(Z)$.

   (b) Find the autocorrelation function for y_t. Does the autocovariance sequence γ_r tend to zero as $r \to \infty$?

   (c) Assuming the spectral distribution function $S_y(\lambda)$ is a step function, find the jumps at $\lambda = \theta$ and $\lambda = -\theta$. Plot $S_y(\lambda)$ under the normalization $S_y(0) = 0$.

5. Let $\{x_t\}$ and $\{y_t\}$ be independent zero-mean covariance-stationary processes with absolutely summable autocovariance functions $\gamma_x(r)$ and $\gamma_y(r)$. Let $s_x(\lambda)$ and $s_y(\lambda)$ be their univariate spectral density functions. Define $z_t = x_t y_t$. Let $\gamma_z(r)$ be the autocovariance function for $\{z_t\}$ and let $s_z(\lambda)$ be its spectral density function.

   (a) Find an expression for $\gamma_z(r)$ in terms of $\gamma_x(r)$ and $\gamma_y(r)$.

   (b) Show that the asymptotic variance of $T^{-1/2}\sum_{t=1}^{T} z_t$ is equal to $2\pi s_z(0)$.

   (c) Using the fact that $\int_{-\pi}^{\pi} e^{-i\lambda k} d\lambda$ is equal to zero when k is a nonzero integer, show that

   $$s_z(0) = \int_{-\pi}^{\pi} s_x(\lambda)s_y(\lambda)d\lambda.$$

   [Hint: start with the right-hand side.]

   (d) Suppose you observe T successive observations on $(x_t,\, y_t)$. Suggest two alternative ways to use periodogram values to estimate the variance of $T^{-1/2}\sum_{t=1}^{T} z_t$.