The Folk Theorem for Repeated Games with Perfect Monitoring

Some Definitions:

Set of feasible payoffs:

$$\mathcal{V} = \text{co}\,\{(g_1(\bar{a}), g_2(\bar{a}), \ldots g_N(\bar{a})) \mid \bar{a} \in \bar{\mathcal{A}}\}.$$

A pure strategy minmax payoff of player i is given by

$$\underline{v}_i = \min_{a_{-i} \in \mathcal{A}^{-i}} \max_{a_i \in \mathcal{A}^i} g_i(a_i, a_{-i}).$$

Set of feasible and individually rational payoffs

$$\mathcal{V}^* = \{v \in \mathcal{V} \mid v_i \geq \underline{v}_i \text{ for all } i\}.$$

Set of payoff vectors achievable in a SPE of a repeated game with discount factor δ : $V(\delta)$

Fact: $V(\delta) \subseteq \mathcal{V}^*$ for all $\delta \in [0, 1)$.

The Folk Theorem for 2-player Games

Theorem. Suppose that there are two players. For any $v \in \mathcal{V}$ with $v_i > \underline{v}_i$ there is a discount factor $\underline{\delta} < 1$, such that for all $\delta \in [\underline{\delta}, 1)$ there is a SPE of the repeated game with (average per period) payoffs v.

Note 1: Average payoffs are defined by

$$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}g_1(A_t),$$

where $(1-\delta)$ is a normalizing factor that makes payoffs of the repeated game be on the same scale as the payoffs of a stage game.

Note 2: The following proof is valid for pure strategy SPE and a pure strategy minmax. See Fudenberg and Tirole or Fudenberg and Maskin (1986) for mixed strategy SPE, and more than two players.

Proof of The Folk Theorem

Proof (Fudenberg and Maskin, 1986): Let $(s_1, s_2) \in \Delta(\bar{A})$ be a probability distribution over pairs of actions that achieves payoff pair v in expectation. Let M_1 be player 1's minmax strategy against player 2, and M_2 be player 2's minmax strategy against player 1.

We would like to construct a SPE for discount factors δ sufficiently close to 1, which achieves payoff vector v. The SPE will be based on two regimes:

- collusive regime, in which players choose (s_1, s_2) according to a public randomizing device

- a punishment regime, in which players choose (M_1, M_2) for T periods

The transitions are as follows: start in the collusive regime. If nobody deviates, stay. If anybody deviates, go to the punishment regime.

In the punishment regime: return to the collusive regime after T periods if nobody deviates. If anybody deviates, restart the punishment regime.

Proof of The Folk Theorem (continued)

To summarize:

- collusive regime, in which players choose (s_1, s_2) according to a public randomizing device
- a punishment regime, in which players choose (M_1, M_2) for T periods

We would like to show that if T is sufficiently large, then we have a SPE for δ sufficiently close to 1. We need the following two conditions to hold:

1. One-period deviation in the collusive regime is not profitable:

$$\underbrace{G_i}_{\text{deviation gain}} + \underbrace{(\delta + \ldots \delta^T)}_{\approx T}(g_i(M_1, M_2) - v_i) \leq 0$$

This condition holds if $T > \frac{G_i}{v_i - g_i(M_1, M_2)}$ and δ is sufficiently close to 1 so $\delta + \ldots \delta^T$ is sufficiently close to T.

2. One-period deviation in the punishment regime is not profitable:

$$\underline{v}_i + (\delta + \ldots \delta^T) g_i(M_1, M_2) + \delta^{T+1} v_i + \ldots \leq$$
$$(1 + \delta + \ldots \delta^{T-1}) g_i(M_1, M_2) + \delta^T v_i + \delta^{T+1} v_i + \ldots$$

This condition holds for all δ sufficiently close to 1, when

$$\underline{v}_i \leq \underbrace{(1 - \delta^T) g_i(M_1, M_2)}_{\to 0} + \underbrace{\delta^T v_i}_{\to \underline{v}_i}$$

QED

Repeated Games Between Long-Lived and Myopic Players, Commitment

Applications:

- optimal government policy (taxation, etc.)

- optimal policy in microeconomics (commitment to quality standards)

The Model: One long-lived player, who cares about a discounted sum of payoffs, faces a series of myopic players, who care about payoffs for only one period

Two interpretations:

- each myopic player is different, e.g. a new customer in dynamic monopoly, or a new potential entrant in a chainstore game

- there is a population (continuum) of myopic players, who are infinitely lived, but only the large long-lived player only observes population average. Then individuals will act as if they maximize their one-period payoffs, because individual deviations do not affect population average.

A Simple Model (Repeated Game with a Long-Lived and a Myopic Player).

A 2-player stage game G, e.g. a simultaneous-move game

$$G = \{N = \{1, 2\}, \ (\mathcal{A}^i)_{i=1,2}, \ (g_i)_{i=1,2}\}.$$

or an extensive-form game.

Repeated game: a long-lived with payoff

$$\sum_{t=1}^{\infty} \delta^{t-1} g_1(A_t)$$

a short-lived player for each period t with payoff

$$g_2(A_t)$$

Assume that short-lived players observe all actions taken in all previous periods.

Example 1

Long-lived seller. Short-lived buyers. In each period, a buyer decides to buy or not. If the buyer decides to buy, the seller decides to produce high or low quality.

Payoffs are as follows:

(0, 0) if no purchase
(1, 1) if purchase, high quality
(2, -1) if purchase, low quality

What happens in a SPE of a stage game?

For what discount factors is there a SPE of the repeated game where the seller produces high quality in every period?

Example 2

Long-lived government. Myopic population. In each period, first the government announces a tax rate $\tau \in [0,1]$. Next, each individual in the population chooses a production level $a \in [0,\infty)$, and obtains output $f(a)$. Assume that $f(0) > 0$, $f'(0) > 1$, $f''(0) < 0$ and $f'(a) \to 0$ as $a \to \infty$. After the output has been realized, the government chooses taxes output at any rate $\tau' \in [0,1]$, and spends the revenues on a public good. Each individual acts to maximize $(1-E[\tau])f(a)-a$, where $E[\tau]$ is the expected tax rate, because he assumes that the aggregate tax revenues and public good spending will not depend on his individual decision. The government cares about the welfare of the population, i.e. its per-period payoff is $g(\tau' f(a)) + (1-\tau')f(a) - a$, where g is the utility that each individual gets from the consumption of the public good. Assume that $g(0) = 0$, $g' > 0$, $g'' < 0$, $g'(0) > 1$, $g'(f(0)) < 1$.

(a) What conditions determine a socially optimal level of output and a socially optimal tax?

(b) What conditions determine a subgame perfect equilibrium of a one-shot game?

(c) Is the subgame perfect equilibrium of the one-shot game socially optimal?

The Set of Subgame Perfect Equilibrium Payoffs

For this characterization, consider a simultaneous-move stage game G.

The set of symmetric SPE payoffs of the long-lived player is characterized by an interval $[\underline{w}, \bar{w}]$ such that

$$\bar{w} = \max_{a_1} g_1(a_1, \widehat{a}(a_1)) + \delta \bar{w}$$

$$\text{s.t.} \quad \forall a_1' \in \mathcal{A}^1, \quad g_1(a_1, \widehat{a}(a_1)) + \delta \bar{w} \geq g_1(a_1', \widehat{a}(a_1)) + \delta \underline{w} \tag{1}$$

where $\widehat{a}(a_1)$ denotes a static best response of the myopic players to a_1.

$$\underline{w} = \min_{a,w} g_i(a_1, \widehat{a}(a_1)) + \delta w$$

$$\text{s.t.} \quad \forall a_1' \in \mathcal{A}^1, \quad g_1(a_1, \widehat{a}(a_1)) + \delta w \geq g_1(a_1', \widehat{a}(a_1)) + \delta \underline{w} \tag{2}$$