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Economics 136
Solutions to Problem Set #2

1. (a) We know that both assets have the same expected return and variance. That is, $E(r_A) = E(r_B) = \mu$ and $\sigma_A^2 = \sigma_B^2 = \sigma^2$. We also know that the asset's returns are uncorellated, i.e. $Cov(r_A, r_B) = 0$. The first step is to recognize that no matter how you divide your wealth between these two assets, your expected return will be the same. Letting w be the weight on asset A, we see that the expected return on your portfolio is

$$E\left(wr_A + (1-w)r_B\right) = E(wr_A) + E\left((1-w)r_B\right) = wE(r_A) + (1-w)E(r_B) = w\mu + (1-w)\mu = \mu.$$

Thus, the only possible reason that you might care about how your wealth is divided between the assets **in this example** is if the risk of the portfolio is different. Specifically, if you are risk-averse a lower variance is preferred, all other things equal (e.g. expected return). Calculate the variance of your portfolio's return as follows:

$$\sigma_p^2 = Var\left(wr_A + (1-w)r_B\right) = w^2\sigma_A^2 + (1-w)^2\sigma_B^2 = \sigma^2\left(w^2 + (1-w)^2\right) = \sigma^2\left(2w(w-1) + 1\right).$$

If $w = 0$ or 1, then all of your wealth is in asset B or asset A, respectively, and we see from the above formula that $\sigma_p^2 = \sigma^2$. However, if $0 < w < 1$, then $2w(w - 1) < 0$, and we see from the above formula that $\sigma_p^2 < \sigma^2$. Intuitively, it pays to diversify—don't put all your eggs in one basket. Since A and B are not correlated, there are times when a bad return for one will be offset by a good return for the other. This lowers the variance of your return. Thus, you should fire your broker!

(b) Differentiate the above expression for the variance of your portfolio's return with respect to w and set it equal to zero to find the optimal weight on asset A:

$$\frac{d}{dw}\left(\sigma^2[w^2 + (1-w)^2]\right) = \sigma^2\left(2w - 2(1-w)\right) = 0 \Rightarrow 4w - 2 = 0 \Rightarrow w^* = 1/2.$$

You should think about why ½ makes sense. What if the assets were correlated, or had different risks. How might that change your answer here?

2. (a) Letting w be the weight on the risky asset, we calculate the portfolio's expected return and variance of return:

$$E(r_p) = E\left(wr_s + (1-w)r_f\right) = wE(r_s) + (1-w)r_f$$
$$\sigma_p^2 = Var\left(wr_s + (1-w)r_f\right) = w^2\sigma_s^2$$

Substitute these into the utility function and differentiate with respect to w. You should get the following first order condition: $E(r_s) - r_f - 2Aw\sigma_s^2 = 0 \Rightarrow w^* = \dfrac{E(r_s) - r_f}{2A\sigma_s^2}$

This is the necessary condition for the weight we put on the risky asset, w, to maximize utility. You might want to also verify it is sufficient (check second order conditions) so that we have indeed found our optimal weight.

(b) Looking at the expression above for w^*, we see that the optimal share of wealth to be invested in the risky asset is proportional to the return premium offered on the risky asset, $E(r_s) - r_f$. Moreover, the proportion depends on the degree of risk aversion, A. Notice in particular that, all other things equal, the investor wants to put more wealth into the risky asset the greater is the premium. Similarly, all else equal, the investor wants to put more weight on the risky asset the smaller is A—i.e. the less risk averse she is. As a further note, consider the fact that we can interpret the single risky asset in this set up as a *collection* of risky assets. For instance, the risky asset here could be a portfolio of all risky assets traded in the market, what we might call the **market portfolio**.

(c) Simply substitute the given values into the expression we found for w: $w^* = \frac{.2-.05}{(2/5)(.5)^2} = 1.5$

This says our investor wants to borrow 50% of her wealth at the risk-free rate and put everything into the risky asset. This is sensible in that is merely describes what the investor wants to do *if* she can borrow at the risk-free rate.

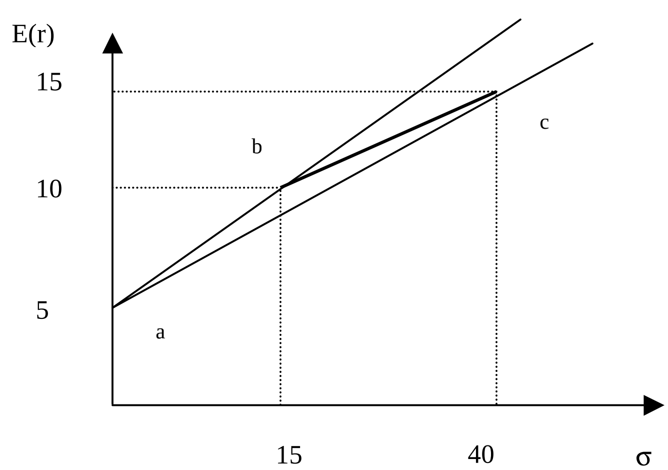
3. (a) Portfolio D dominate E and H, so E & H are inefficient; 1/3A + 2/3D dominates C, so C is inefficient; 5/9 A + 4/9 D dominates B, so B is inefficient; 7/9 D + 2/9 A dominates F, so F is inefficient. A, D, and G are all efficient, since they are not dominated by any portfolio.

(b) A portfolio is fully diversified if it has only systematic or market risk, which will be true if the portfolio lies on the capital market line. If A is the market portfolio, then portfolios A, D, and G all lie on the CML and all are fully diversified. D has a beta of 0 since it is uncorrelated with the market portfolio. A has a beta of 1 since it is perfectly correlated with itself. Finally G has a beta of 2, calculated as (21 -3)/(12 - 3).

(c) The beta of portfolio B is calculated as $\frac{E(r_B)-r_f}{E(r_m)-r_f} = \frac{8-3}{12-3} = \frac{5}{9}$. B does not lie on the CML, but is does lie on the SML, as all securities do in equilibrium.

4. (a) The efficient frontier of risky assets *only* is the line segment *bc* in the graph below. In this example no portfolio of risky assets is dominated by another portfolio of risky assets, so every portfolio of risky assets in on the efficient frontier of risky assets.

(b) We can graph all the possible ways that the risk free asset can be combined with a portfolio of risky assets as follows. Suppose we just combine assets A and C only. Since A can be sold short, the line starting at point *a* and passing through point *c* shows all the portfolios of A and C that are possible. Similarly, any line starting from point *a* that passes through a portfolio of risky assets--a point on the efficient frontier of risky assets, segment *bc*--will show feasible combinations of the risk free with a risky portfolio. In particular, the line from point *a* passing through point *b* are feasible combinations, which we get by combining assets A and B in various proportions. We can see by inspection that only by combining A and B do we get portfolios that are not dominated in this example. Thus, the ray from point *a* passing through *b* will be our new efficient frontier.



(c) We can see from the graph above that any portfolio with asset C in it is dominated, so an investor will never hold asset C. Thus, this cannot be an equilibrium. Over time, the price of C should fall causing its expected return to rise. In general, the whole distribution of returns may change until we get a single straight line whereby an investor is indifferent as to which portfolio of risky assets he holds.

(d) If short selling of the risky asset were allowed, then the efficient frontier of risky assets can extend in both directions. Specifically, we could form a risk free portfolio *with risky assets alone*. To see this, remember that the standard deviation of a portfolio of the two perfectly positively correlated assets is $\sigma_p = w\sigma_B + (1-w)\,\sigma_C$. With short selling allowed, for example, we could have $w = 1.6$ and $1 - w = -.6$ This would mean the investor is selling short asset C in an amount equal to 60% of her wealth. The result would be a portfolio of B and C with a standard deviation of $\sigma_p = (1.6)(.15) - (.6)(.4) = 0$, and expected return of $E(r_p) = 1.6(.10) - .6(.15) = .07$. This shows that we can construct a risk free asset with a 7% return from two perfectly positively correlated assets through short selling. The trick is that the short position will be *perfectly negatively* correlated with the other asset, and we know it is always possible to construct a risk free portfolio from to two such assets. We are still led to the conclusion that the asset markets are not in equilibrium, although this time we can spin a new twist. The presence of two different risk free rates presents an *arbitrage* opportunity, which is an investment strategy that provides you with a payoff without having to invest any of your own money. Here we simply borrow at 5% and invest in the risk free portfolio earning 7%--a sure way to get rich. Such a divergence in risk-free rates cannot last long, so we are not in equilibrium.

5. The formula we will use to price the stock is $P_t = \dfrac{E(d_{t+1})}{E(r) - g}$. The numerator is the dividend that ACME is expected to pay next period, which we are told is \$4. *g* is the expected constant growth rate of dividends, we are also told is 7%. E(r) is the required rate of return on a security of ACME's risk class, which we must calculate. We know that ACME has a beta of 2, and that the excess return on the market is 10% - 3% = 7%. We can use the SML to calculate ACME's required return to be
$E(r) = 3\% + [7\%]2 = 17\%$. Thus we have $P_t = \$4/(.17 - .07) = \$4/.1 = \$40$.