What does $q$ represent? The general solution to a differential equation such
as (2) is a forward-looking integral expression (as you can verify by differentiating). ${ }^{1}$ (All of this is true even if we allow the interest rate $r$ to vary over time.) The general solution (for a constant interest rate) is

$$
q(t)=\int_{t}^{\infty} \mathrm{e}^{-r t}\left[A F_{K}(K(s), L(s))+\frac{\chi}{2}\left(\frac{I(s)}{K(s)}\right)^{2}\right] \mathrm{d} s+b \mathrm{e}^{r t}
$$

where I have made the substitution

$$
\frac{(q-1)^{2}}{2 \chi}=\frac{\chi}{2}\left(\frac{I}{K}\right)^{2}
$$

and $b$ is an arbitrary constant. The economically relevant solution imposes the transversality condition

$$
\lim _{t \rightarrow \infty} \mathrm{e}^{-r t} q(t) K(t)=0
$$

which obliges us to set $b=0$ above. In that case

$$
q(t)=\int_{t}^{\infty} \mathrm{e}^{-r t}\left[A F_{K}(K(s), L(s))+\frac{\chi}{2}\left(\frac{I(s)}{K(s)}\right)^{2}\right] \mathrm{d} s
$$

${ }^{1}$ Use the following fact from calculus. Let

$$
f(t)=\int_{a(t)}^{b(t)} g(s, t) \mathrm{d} s
$$

Then

$$
f^{\prime}(t)=g[b(t), t] b^{\prime}(t)-g[a(t), t] a^{\prime}(t)+\int_{a(t)}^{b(t)} \frac{\partial g(s, t)}{\partial t} \mathrm{~d} s
$$

which means that the shadow value of a unit of installed capital equals the discounted future marginal products, plus the future contributions to lowering the installation costs of optimal investments (it is cheaper at the margin to add capital to a larger pre-existing capital stock).

The $q$ variable defined above is marginal $q$, the shadow value of an extra unit of capital, given $K$. Empirical work on investment, however, does not have access to this variable: researchers must use as a proxy stock-market value divided by total capital-in-place, $\Pi / K$, which amounts to average $q$. What is the relationship between average and marginal $q$ ? This was clarified in a famous 1982 article in Econometrica by Fumio Hayashi.

Notice that

$$
\begin{aligned}
\frac{\mathrm{d}(q K)}{\mathrm{d} t} & =q \dot{K}+\dot{q} K \\
& =r q K-\left(A F_{K} K+\frac{\chi I^{2}}{2 K}\right)+q I \\
& =r q K-\left[A F(K, L)-w L+\chi \frac{I^{2}}{2 K}\right]+I\left(1+\chi \frac{I}{K}\right) \\
& =r(q K)-\left[A F(K, L)-w L-I-\chi \frac{I^{2}}{2 K}\right]
\end{aligned}
$$

Imposing the transversality condition, we can integrate forward (solving for the composite variable $q K$ ) to conclude that

$$
q(t) K(t)=\int_{t}^{\infty} \mathrm{e}^{-r(s-t)}\left[A(s) F(K(s), L(s))-w(s) L(s)-I(s)-\frac{\chi}{2}\left(I(s)^{2} / K(s)\right)\right] \mathrm{d} s=\Pi(t)
$$

We see that marginal $q$ and average $q$ are equal:

$$
q=\frac{\Pi}{K}
$$

The key facts used to derive this prediction are that markets are competitive and that the function $\psi(K, L, I)=F(K, L)-\frac{\chi I^{2}}{2 K}$ displays constant returns to scale, i.e., for any nonnegative number $\lambda, \psi(\lambda K, \lambda L, \lambda I)=\lambda \psi(K, L, I)$.

