Economics 230a **Fall 2011**

Production, Expenditure, and Indirect Utility Functions

We will often use a general version of the production function, $F(\boldsymbol{x}) \leq 0$, where elements of the vector $\boldsymbol{x}$ may be either positive or negative, with positive values representing outputs and negative values representing inputs. When production is based on a constant returns to scale technology $F(\cdot)$ must be homogeneous of degree zero, i.e., $F(\lambda\boldsymbol{x}) = F(\boldsymbol{x})$ for any scalar λ, since $\lambda\boldsymbol{x}$ will be a feasible vector of inputs and outputs if and only if $\boldsymbol{x}$ is also.

Written this way, the production function encompasses production functions relating outputs to inputs. For example, production with a single output, y, and vector of inputs, $\boldsymbol{z} = (z_1, z_2, ...)$ described by the relationship $y \leq G(\boldsymbol{z})$ may also be written $y - G(\boldsymbol{z}) \leq 0$ or $x_1 - G(-x_2, -x_3, ...) = F(\boldsymbol{x}) \leq 0$, where $x_1 = y$ and $x_i = -z_{i-1}$ for $i > 1$. Note that if $G(\cdot)$ is homogeneous of degree 1 then $F(\cdot)$ as defined based on $G(\cdot)$ will be homogeneous of degree 0.

We use the function $F(\cdot)$ rather than $G(\cdot)$ because it is more general, covering cases where there may be multiple outputs and inputs. But the usual properties apply, in particular that competitive firms will set the ratio $F_i/F_j = p_i/p_j$, i.e., that the marginal rate of transformation between any two elements of production equals the relative price between these two elements. When element i is an output and j is an input, we may interpret this as setting the marginal revenue product p_iF_j/F_i equal to the factor price p_j. When elements i and j are both inputs, $F_i/F_j = p_i/p_j$ says that the ratio of marginal products should equal the ratio of factor prices. When both elements are outputs, then the relationship says that relative output prices should equal the ratio of marginal costs.

The expenditure function defines the minimum expenditure needed to achieve a given level of utility for a given price vector, or $E(\boldsymbol{p}, u)$. Likewise, the indirect utility function is the maximum utility achievable for a given level of income and a given price vector, or $V(\boldsymbol{p}, y)$. Among the properties that we will find useful are:

> $\frac{\partial E}{\partial p_i} = x_i^c(\boldsymbol{p}, u)$, where $x_i^c(\cdot)$ is the compensated demand for good i. This is Shepard's lemma, and says that the first-order income effect of the increase in a good's price is the amount of that good purchased.
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> $E(\boldsymbol{p}, V(\boldsymbol{p}, y)) = y$; that is, since $V(\boldsymbol{p}, y)$ is the maximum utility achievable at prices $\boldsymbol{p}$ and income available for expenditure y, then the amount of income needed to achieve this level of utility when the price vector actually is $\boldsymbol{p}$ is y.
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> $x_i^c\big(\boldsymbol{p}, V(\boldsymbol{p}, y)\big) = x_i(\boldsymbol{p}, y)$; that is, the compensated demand at a given price vector and the utility level achievable at that price vector and a given level of income equals the ordinary demand at the same price vector and level of income.
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> $\frac{\partial V/\partial p_i}{\partial V/\partial y} = -\frac{\partial E}{\partial p_i}$, which follows from the implicit function theorem. By Shepard's lemma, this means that $\frac{\partial V}{\partial p_i} = -\frac{\partial V}{\partial y} x_i^c(\boldsymbol{p}, u) = -\frac{\partial V}{\partial y} x_i^c(\boldsymbol{p}, V(\boldsymbol{p}, \mathrm{y})) \ = \ -\lambda x_i(\boldsymbol{p}, y)$, where we define λ as the marginal utility of income.