Derivation of the User Cost of Capital

Consider a firm wishing to maximize its value at date t,

$$(1) \qquad V_t = \int_t^{\infty} e^{-r(s-t)} X_s ds \, ,$$

where r is the discount rate that applies to the corporation's real activities and X_s is the firm's cash flow at date s from these activities,

$$(2) \qquad X_s = (1-\tau_s) p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_{-\infty}^{s} D_u (s-u) q_u I_u du \, .$$

In (2), p_s is the price of output, K_s is the capital stock that is the single argument in the production function $F(\cdot)$, q_s is the price of capital goods, and I_s is the flow of investment. The corporate tax system has three components: τ_s is the corporate tax rate at date s, k_s is the initial subsidy to investment (e.g., an investment tax credit), and $D_u(s-u)$ is the depreciation deduction at date s per dollar of investment made at an earlier date u. This deduction depends not only on the age of the asset, $(s-u)$, but also on the tax depreciation rules that prevailed at date u.

Inserting (2) into (1) yields:

$$\begin{aligned} V_t &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s) p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_{-\infty}^{s} D_u (s-u) q_u I_u du \right) ds \\ (3) \qquad &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s) p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_t^{s} D_u (s-u) q_u I_u du + \tau_s \int_{-\infty}^{t} D_u (s-u) q_u I_u du \right) ds \\ &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s) p_s F(K_s) - q_s I_s (1-k_s) + \tau_s \int_t^{s} D_u (s-u) q_u I_u du \right) ds + \overline{V}_t \, , \end{aligned}$$

where the second line of (3) breaks the flows of depreciation allowances into two pieces: those attributable to investment after date t and before date t. The second piece, $\overline{V}_t$, affects the value of the firm at date t, but not its decisions from date t onward, and so may be ignored in the optimization process. (It will, however, be relevant when we consider the issue of corporate tax incidence.) The remaining expression for firm value can be simplified by changing the order of integration for depreciation allowances (first over date of allowances, then over date of investment, rather than starting with date of investment), leading to:

$$\begin{aligned} V_t &= \int_t^{\infty} e^{-r(s-t)} \left((1-\tau_s) p_s F(K_s) - q_s I_s (1-k_s) + q_s I_s \int_s^{\infty} e^{-r(u-s)} \tau_u D_s (u-s) du \right) ds + \overline{V}_t \\ (4) \qquad &= \int_t^{\infty} e^{-r(s-t)} \big((1-\tau_s) p_s F(K_s) - q_s I_s (1-\Gamma_s) \big) ds + \overline{V}_t \, , \end{aligned}$$

where $\Gamma_s = k_s + \int_s^{\infty} e^{-r(u-s)} \tau_u D_s (u-s) du$ is the present value of tax benefits per dollar invested at s.

The firm seeks to maximize its value at time t, as defined in expression (4). Determining the optimal investment policy requires further specification of the firm's technology. It is standard to assume that capital depreciates exponentially at rate δ, that is:

(5) $$\dot{K}_t = I_t - \delta K_t .$$

Note that δ is capital's rate of actual, or *economic* depreciation, and is generally distinct from the pattern of depreciation allowances specified by the function $D(\cdot)$ defined above.

Inserting (5) into (4), one can then solve for the optimal capital stock path using the calculus of variations. The Euler equation, $\frac{\partial V_t}{\partial K_s} - \frac{d\left(\partial V_t / \partial \dot{K}_s\right)}{ds} = 0$, yields the following expression for the optimal capital stock at date s:

(6) $$F'(K_s) = \frac{q_s^*}{p_s} \frac{\left(r + \delta - \dot{q}_s^* / q_s^*\right)}{(1 - \tau_s)},$$

where $q_s^* = q_s(1 - \Gamma_s)$, which one may think of as the effective price of capital goods, taking into account the present value of tax benefits directly associated with investment. The expression on the right-hand side of (6), the implicit rental price of capital, is commonly referred to as the *user cost of capital*. With a constant tax system, $\dot{q}_s^* / q_s^*$ is just $\dot{q}_s / q_s$ and the term in parentheses in the numerator is just the real required return to investors $r - \dot{q}_s / q_s$ plus the rate of depreciation, δ.

<u>Special Cases</u> (with tax parameters constant over time):

Immediate expensing: $\Gamma_s = \tau_s$, so the user cost becomes $\frac{q_s}{p_s}\left(r + \delta - \dot{q}_s / q_s\right)$; so the tax system affects investment only through its impact on the required rate of return, r. Note that, in this case, $\bar{V} = 0$, so that the value of the firm will equal $(1-\tau)q_s K_s$.

Economic depreciation allowances (at replacement cost): $D_s(u - s) = \frac{q_u}{q_s} \delta e^{-\delta(u-s)}$; for a constant inflation rate, this implies that $\Gamma_s = \tau \frac{\delta}{r + \delta - \dot{q}/q}$, so the user cost becomes $\frac{q_s}{p_s}\left(\frac{r - \dot{q}_s / q_s}{1 - \tau} + \delta\right)$. The tax system effectively taxes the net (after depreciation) return to investment, $r - \dot{q}_s / q_s$. In this case, new and old capital receive the same present value of depreciation allowances per unit of capital, so the new and old capital have the same market value per unit. This implies that the value of the firm will equal $q_s K_s$.