Economics 101A
Section Notes
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Nash Equilibrium and Duopoly Theory

1 Nash Equilibrium

Consider the case where the case with $N = 2$ firms, indexed by $i = 1, 2$. Most of what we consider here is generalizable for larger N (general oligopoly) but working with 2 firms makes things much easier. Let firm 1's profit depend its own action a_1 ("action" is defined very broadly here) as well as firm 2's action, so we can write $\pi_1 (a_1, a_2)$, and similarly for firm 2 $\pi_2 (a_1, a_2)$. The general ideas here would also be applicable if we used utility, e.g. $U_1 (a_1, a_2)$, rather than profits.

1.1 Definition

A set of actions (a_1^N, a_2^N) constitutes a *Nash equilibrium* iff

$$\begin{aligned} \pi_1 \left(a_1^N, a_2^N\right) &\geq \pi_1 \left(a_1, a_2^N\right) \quad \text{for all } a_1\text{, and} \\ \pi_2 \left(a_1^N, a_2^N\right) &\geq \pi_1 \left(a_1^N, a_2\right) \quad \text{for all } a_2 \end{aligned}$$

In other words a set of actions is a Nash equilibrium if each firm cannot do better for itself playing its Nash equilibrium action given other firms play their Nash equilibrium action.

1.2 Solving for Nash Equilibria

To find the Nash equilibrium, we should consider each firm's profit maximization problem where each firm takes each other's action as given parametrically, but which are resolved simultaneously:

$$\max_{a_1} \pi_1 (a_1, a_2) \quad \text{and} \quad \max_{a_2} \pi_2 (a_1, a_2)$$

. Oftentimes finding a Nash involves checking all the possible combinations (a_1, a_2) and asking yourself "is this a Nash equilibrium?" Sometimes it is possible to eliminate dominated actions iteratively (see a book on game theory) to narrow the cases that need to be checked. However, assuming profit functions are continuously differentiable, concave, and a_1^N and a_2^N are both positive, we can take first order conditions. The separate first order conditions for each firm is just

$$\frac{\partial \pi_1 \left(a_1^N, a_2^N\right)}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial \pi_2 \left(a_1^N, a_2^N\right)}{\partial a_2} = 0 \qquad \text{(Nash FOC)}$$

which is a system of 2 equations in 2 unknowns a_1^N, a_2^N, and so usually a little algebra will yield the solution. Additionally, the second order conditions imply that the profit functions should be concave in each firm's own action, namely

$$\frac{\partial^2 \pi_1 \left(a_1^N, a_2^N\right)}{\partial a_1^2} = 0 \quad \text{and} \quad \frac{\partial^2 \pi_2 \left(a_1^N, a_2^N\right)}{\partial a_2^2} = 0 \qquad \text{(Nash SOC)}$$

The second order conditions say that marginal profits should slope downwards with respect to the firm's own action.

1.3 Best Response Functions

By the implicit function theorem the FOC for firm 1 alone defines what it will play given a_2 , i.e. firm 1's **best response function** (or *reaction curve*) $a_1 = r_1 (a_2)$. By solving the FOC for firm for a_1 in terms of a_2 we can find the best response function as the FOC

$$\frac{\partial \pi_1 (r_1 (a_2), a_2)}{\partial a_{11}} = 0$$

holds for all a_2 (not just the Nash). Geometrically the best response cutves can be seen as finding a point in the (a_1, a_2) where the firm's isoprofit curve, which has slope

$$-\frac{\partial \pi_1 / \partial a_1}{\partial \pi_2 / \partial a_2}$$

is tangent to the horizontal line defined by a_2, which has slope zero as firm 1 takes firm 2's action as fixed as if it were a constraint. [1] This can be interpreted as firm 1 solving for its maximum profit subject to the constraint that a_2 is a fixed number. A similar best response function $r_2(a_1)$ can be defined for firm 2, where

$$\frac{\partial \pi_2 \left(r_2(a_1), a_1\right)}{\partial a_2} = 0$$

The best response curve for firm 2 lies where the isoprofit curve for firm 2 is vertical as it is tangent to the line for a fixed value of a_1. A Nash equilibrium can be seen as where each action is a best response to the other firm's action

$$a_1^N = r_1\left(a_2^N\right) \quad \text{and} \quad a_2^N = r_2\left(a_1^N\right)$$

This is where the best response curves cross in a graph with a_1 on one axis and a_2 on the other. Substituting in one best response function into the other gives

$$a_1^N = r_1\left(r_2\left(a_1^N\right)\right)$$

which states that firm 1's action a_1^N is its best response to firm 2's best response to firm 1's action. Thus, the Nash equilibrium concept produces a consistent set of responses. Firm 1 cannot do better given that firm 2 is using its best response to firm 1.

1.4 Strategic Complements and Substitutes

It is useful to know how one firm will react if the other firm changes its action. Cases where a greater action by 2 elicits more of a response by 1, i.e. $\frac{dr_1}{da_2} > 0$, identifies a situation where a_1 and a_2 are called **strategic complements**. The alternate case where $\frac{dr_1}{da_2} < 0$, is where a_1 and a_2 are called **strategic substitutes**. Recall that the equation $\frac{\partial \pi_1(r_1(a_2), a_2)}{\partial a_1} = 0$ holds for all a_2 and therefore we can differentiate this expression with respect to a_2 to get

$$\frac{d}{da_2}\left[\frac{\partial \pi_1\left(r_1(a_2), a_2\right)}{\partial a_1}\right] = \frac{\partial^2 \pi_1}{\partial a_1^2}\frac{dr_1}{da_2} + \frac{\partial^2 \pi_1}{\partial a_2 \partial a_1} = 0$$

and so solving for the slope of the best response curve

$$\frac{dr_1}{da_2} = -\left(\frac{\partial^2 \pi_1}{\partial a_1^2}\right)^{-1} \frac{\partial^2 \pi_1}{\partial a_2 \partial a_1}$$

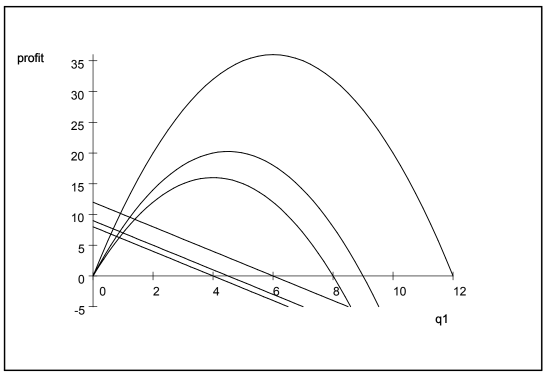
The sign of this expression depends on the sign of the second derivatives of the profit function. The sign of $\frac{\partial^2 \pi_1}{\partial a_1^2}$ is negative because of (Nash SOC), i.e. the marginal profit function is downward sloping in a_1. Therefore we can conclude that the sign of $\frac{dr_1}{da_2}$ will have the same sign as the cross partial derivative $\frac{\partial^2 \pi_1}{\partial a_2 \partial a_1}$, i.e.

$$sign\left(\frac{dr_1}{da_2}\right) = sign\left(\frac{\partial^2 \pi_1}{\partial a_2 \partial a_1}\right)$$

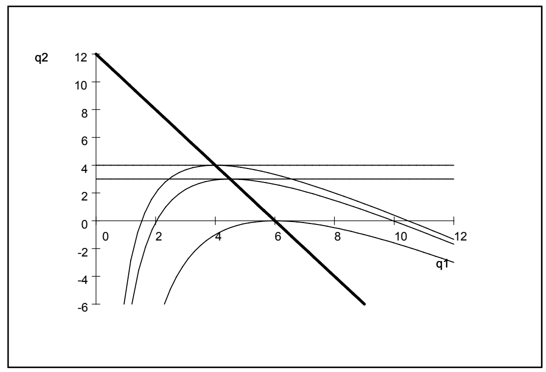
So if firm 2's action a_2 increases the marginal profit of firm 1's own action $\frac{\partial \pi_1}{\partial a_1}$, then firm 1 will increase its action a_1. The following example should help make this clearer.

[1] The isoprofit curve is defined by the equation $\pi_1(a_1, a_2) = k$ where k is a constant. Treating a_2 as a function of a_1 and differentiating totally $\pi_1(a_1, a_2(a_1)) = k$ with respect to a_1 and solving for the implicit derivative gives $\frac{da_2}{da_1} = -\frac{\partial \pi_1 / \partial a_1}{\partial \pi_1 / \partial a_2}$

Example 1 ***Cournot Competition*** *In this case firms compete in quantities* q_1 *and* q_2 *(which are* a_1 *and* a_2 *above). Take the case where inverse demand is given by* $p = 12 - q_1 - q_2$ *and costs are zero. So* $\pi_1(q_1, q_2) = (12 - q_1 - q_2)\, q_1$ *and* $\pi_2(q_1, q_2) = (12 - q_1 - q_2)\, q_2$. *The first order conditions for firms 1 and 2 are* $\partial\pi_1/\partial q_1 = 12 - 2q_1 - q_2 = 0$ *and* $\partial\pi_2/\partial q_2 = 12 - 2q_2 - q_1 = 0$. *Solving for the reaction functions we get* $r_1(q_2) = 6 - q_2/2$ *and* $r_2(q_1) = 6 - q_1/2$. *Below are two graphs which give a graphical derivation of the best response function for firm 1. The first gives the profit functions and marginal profit functions for firm 1 given firm 2 produces zero* $q_2^0 = 0$, *the cartel quantity* $q_1^C = 3$ *(see below), or the Nash quantity* $q_2^N = 4$. *As* q_2 *increases the marginal profit function for firm 1 shifts down, as* $\frac{\partial^2\pi_1}{\partial q_2 \partial q_1} = -1 < 0$. *This shifts firm 1's optimal response* q_1 *to the left as marginal profit is downward sloping, i.e. as* $\frac{\partial^2\pi_1}{\partial q_1^2} = -2$, *implying that* $dr_1/da_2 = -1/2$ *so quantities are strategic substitutes in Cournot competition. The second graph shows how the best response curve of firm 1 is given by the points where the isoprofit curves of firm 1 are perfectly horizontal, i.e. they are tangent to the constraint that* a_2 *is a fixed number. Note that profits are higher on isoprofit curves which are lower and that the monopoly case corresponds to where the isoprofit curve is tangent to the* q_1 *axis where* $q_2 = 0$.

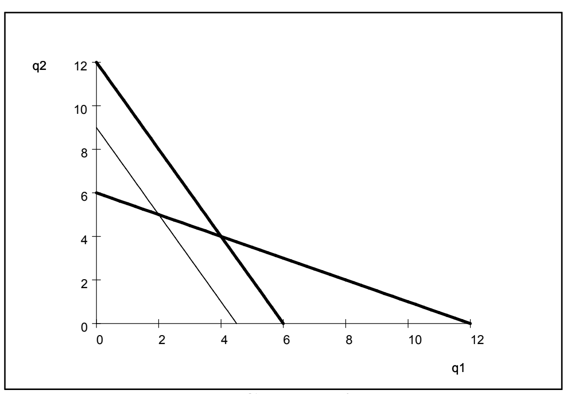


Profit and Marginal Profit of Firm 1



Deriving the Best Response Function

The graphical derivation of the best response curve for firm 2 is similar. The two curves intersect at Cournot-Nash equilibrium quantities $q_1^N = q_2^N = 4$, *as shown below on the left, and each firm makes a profit of* $\pi_1(4,4) = \pi_2(4,4) = 16$. *Now say that firm 1 were to suddenly experience higher costs* $c = 3$ *rather than zero. A little work shows that this would shift firm 1's reaction curve inwards to* $r_1(a_2) = 9/2 - r_2/2$, *causing firm 1's output to fall to* 2, *and because quantities are strategic substitutes, firm 2's output would rise to* 7.

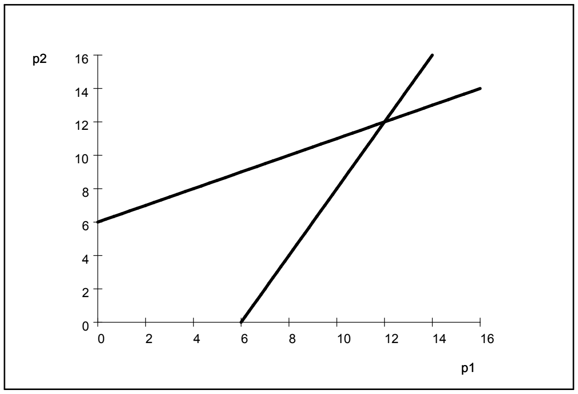


Best Response Curves of Both Firms

Example 2 ***Hotelling Competition*** *Assume that 2 firms sell goods in a partly differentiated market with their demands decreasing in their own price but increasing in their competitors price*[2], *with specific functional*

[2] This specific model would be justified by assuming the two competitors are located on opposite ends of a road with a length

forms given by $q_1 = 12 - p_1 + p_2$ *and* $q_2 = 12 - p_2 + p_1$*. Assuming costs are zero, profits for each firm as a function of prices is then given by* $\pi_1(p_1, p_2) = (12 - p_1 + p_2)\,p_1$ *and* $\pi_2(p_1, p_2) = (12 - p_2 + p_1)\,p_2$*. The FOC for firm 1 is* $\partial\pi_1/\partial p_1 = 12 - 2p_1 + p_2 = 0$ *which yields the best response function* $r_1(p_2) = 6 + p_2/2$*. Similarly the best response function for firm 2 is given by* $r_2(p_1) = 6 + p_1/2$*. The two best response functions are increasing in each others' prices* $dr_1/dp_2 = dr_2/dp_1 = 1/2$*, exhibiting that prices are strategic complements in the Hotelling model. The best response curves intersect at the equilibrium prices* $p_1^N = p_2^N = 12$ *as shown below, leading to profits of* $\pi_1(12, 12) = \pi_2(12, 12) = 144$*.*



Hotelling Best Responses

2 Joint Profit Maximization

Because the firms' profits depend directly on actions they do not control, the firms could jointly raise their profits by coordinating their actions.

2.1 First Order Conditions

Say the firms were to collude to maximize the sum of their profits, solving the following problem

$$\max_{a_1, a_2} \pi_1(a_1, a_2) + \pi_2(a_1, a_2)$$

In this case there are two first order conditions

$$\begin{aligned} \frac{\partial \pi_1}{\partial a_1}(a_1^*, a_2^*) + \frac{\partial \pi_2}{\partial a_1}(a_1^*, a_2^*) &= 0 \\ \frac{\partial \pi_2}{\partial a_2}(a_1^*, a_2^*) + \frac{\partial \pi_1}{\partial a_2}(a_1^*, a_2^*) &= 0 \end{aligned} \qquad \text{(Joint FOC)}$$

2.2 Externalities

The latter derivative in each of these FOC, i.e. $\partial\pi_1/\partial a_2$ and $\partial\pi_2/\partial a_1$ represents the **externality** that each firm has on the other through its action. If the action of firm 2 increases the profits of firm 1, $\partial\pi_1/\partial a_2 > 0$, then firm 2's action is said to have a **positive externality** on firm 1. If instead the action of firm 2 decreases the profits of firm 1, $\partial\pi_1/\partial a_2 < 0$, then firm 2's action is said to have a **negative externality**

of 24 kilometers with consumers uniformly distributed along the road of one per kilometer, each demanding one unit of the good, and incurring a cost of 50 cents per kilometer in travel costs (both ways). In this case where all consumers buy, then the amount sold by firm 1 will be determined by the consumer at distance q_1 who will be indifferent between butying at firm 1 and firm 2. Thus q_1 is determined by the equation

$$p_1 + \frac{q_1}{2} = p_2 + \frac{24 - q_1}{2}$$

which yields the demand function for q_1 and $q_2 = 24 - q_1$.

of firm 1. In the Nash equilibrium, firms ignore the externalities (look at the Nash FOC) yielding a lower joint profit maximization then would be obtained than if they were taken into account. In the case where firms exert positive externalities on each other they would do better by increasing their level of a, while if they exert negative externalities on each other, they would collectively do better by decreasing their level of a. [3]

2.3 Tangency

Note that the (Joint FOC) can be rewritten and combined to show that

$$\frac{\partial\pi_1/\partial a_1}{\partial\pi_2/\partial a_1} = \frac{\partial\pi_1/\partial a_2}{\partial\pi_2/\partial a_2} = -1 \Rightarrow \frac{\partial\pi_1/\partial a_1}{\partial\pi_1/\partial a_2} = \frac{\partial\pi_2/\partial a_1}{\partial\pi_2/\partial a_2}$$

which implies that the isoprofit curves are tangent at the joint profit maximum. The isoprofit curves are not tangent, but perpendicular, at the Nash equilibrium because where the best response curves cross firm 1's isoprofit curve is horizontal while firm 2's isoprofit curve is vertical. This means that there are points on the graph inside both isoprofit curves (below and to the left) where both firms could be better off. However without coordination, these superior points are not an equilibrium as they are off the firms' best response curves. If firm 1 chooses its best response it can increase its profit although its competitor firm 2 will lose more profits than firm 1 will gain.

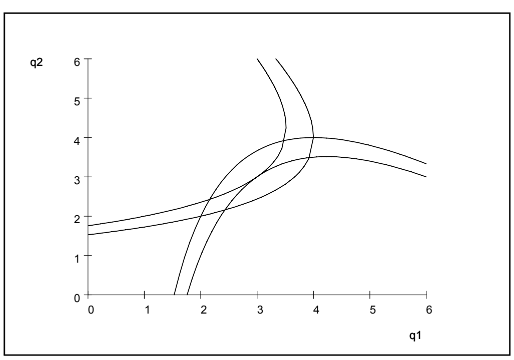
Example 3 ***Cournot Competition*** *Just differentiating the profit functions we can see that if a firm increases its output then it exerts negative externalities on the other as* $\partial\pi_1/\partial q_2 = -q_1 < 0$ *and* $\partial\pi_2/\partial q_1 = -q_2 < 0$ *Say both firms coordinate to maximize the sum of their profits, which is given by*

$$\begin{aligned}\pi_1(q_1,q_2) + \pi_2(q_1,q_2) &= (12 - q_1 - q_2)\,q_1 + (12 - q_1 - q_2)\,q_2 \\ &= (12 - q_1 - q_2)(q_1 + q_2) \\ &= 12(q_1 + q_2) - (q_1 + q_2)^2\end{aligned}$$

The FOC in this case will be identical for both q_1 *and* q_2

$$12 - 2(q_1 + q_2) = 0$$

Therefore we can conclude that $q_1 + q_2 = 6$ *which is the monopoly quantity* q_M*. However this does not say how production should be split between the two firms (not surprising since marginal costs are always the same, zero, at both firms). A typical assumption is to assume that the firms act as a cartel and split production (and profits) equally, so that each firm produces* $q_1^C = q_2^C = 3$*, and earns profits* $\pi_1(3,3) = \pi_2(3,3) = 18$*. The graph below shows a pair of isoprofit curves for each firms. At the cartel solution* $(3,3)$ *two of the curves are tangent; at the Nash equilibrium* $(4,4)$ *two of the curves are perpendicular.*



However firm 1's best response to $q_2 = 3$ *is not to produce* 3 *but to produce* $r_1(3) = 9/2$*. This gives a higher profit to firm 1 of* $\pi_1(9/2,3) = 81/4$*, a gain of* $9/4$*, although firm 2 would then earn a profit of* $\pi_2(9/2,3) = 27/2$*, incurring a loss of* $9/2$ *which is greater than firm 1's gain. Firm 2 would not cooperate with firm 1 if it believed that firm 1 would cheat on the cartel arrangement.*

[3] Of course we are leaving out the effects on consumers. In some cases (like Cournot competition) the joint profit maximizing solution makes firms off but hurts consumers more than it helps firms, as illustrated in the dead weight loss created.

Example 4 ***Hotelling Competition*** *The joint profits of the two firms here are given by*

$$\begin{aligned}\pi_1(p_1,p_2)+\pi_2(p_1,p_2) &= (12-p_1+p_2)\,p_1+(12-p_2+p_1)\,p_2\\ &= 12(p_1+p_2)-p_1^2+2p_1p_2-p_2^2\\ &= 12(p_1+p_2)-(p_1-p_2)^2\end{aligned}$$

In this case profits can be made infinite by choosing equal prices for both $p_1 = p_2$ which makes the second term zero, and then setting the prices arbitrarily high to increase the first term without bound. Of course this is not a realistic conclusion, it comes from the fact that we used a very simple model of Hotelling competition. [4]

3 Sequential Equilibrium

The previous analysis assumed that both firms moved simultaneously. As we will show below, things change considerably if one firm moves before the other in a sequence of two rounds. Assume now that firm 1 moves before firm 2, and that it is able to anticipate firm 1's reaction. These types of situations are often known as **Stackelberg Games.**

3.1 Backward Induction

The optimization technique used to solve for how the firms will behave is known as **backward induction**. This means that we will first solve how firm 2 will act in the second round in response to how firm 1 acts in the first round. Second we move back in time to solve for how firm 1 will act in the first round, using our solution for the second round to model firm 1's anticipation of how firm 2 will react. The technique of backward induction is a fundamental technique in determining how to solve for optimal strategies in a number of game theoretic situations, from tic-toe and chess, to more serious situations.

3.2 Round Two

Firm 2's problem is identical to that found in the Nash equilibrium. It takes a_1 as given and maximizes its profit over a_2, producing a best response function $r_2(a_1)$.

$$\max_{a_2} \pi_2(a_1,a_2)$$

3.3 Round One

\The difference lies in firm 1's problem as it no longer takes a_2 as fixed number but as a best response to what it does, $a_2 = r_2(a_1)$. Thus instead of taking a_2 as a fixed number as its constraint (a horizontal line on the (a_1,a_2) graph) it takes $a_2 = r_2(a_1)$ as its constraint (which is the best response curve on the graph). Substituting in the constraint into firm 1's objective function we have

$$\max_{a_1} \pi_1(a_1, r_2(a_1))$$

Differentiating totally with respect to a_1 yields the first order condition

$$\frac{\partial \pi_1}{\partial a_1}\left(a_1^S, r_2\left(a_1^S\right)\right)+\frac{\partial \pi_1}{\partial a_2}\left(a_1^S, r_2\left(a_1^S\right)\right)\frac{dr_2\left(a_1^S\right)}{da_1}=0 \qquad \text{(Sequential FOC)}$$

Notice that the second term in this FOC does not appear in the (Nash FOC). This equation can be solved for a_1^S provided we know the reaction curve of firm 2 (recall $dr_2/da_1 = -\left(\partial^2\pi_2/\partial a_2^2\right)^{-1}\left(\partial^2\pi_2/\partial a_2\partial a_1\right)$),

[4] The FOC in this case are given by $12-2(p_1-p_2)=0$ and $12+2(p_1-p_2)=0$ which together imply $p_1-p_2=6$ and $p_1-p_2=-6$ which is impossible. The FOC do not help in finding a maximum because no maximum existes (infiniti is not a real maximum). The problem with this model lies in our assumption that all of the people along the road in the Hotelling model buy regardless of price - Hotelling's original model is in fact much richer and allows people to not buy at all if the price is too high.

and a_2^S can be found just by plugging in a_1^S into firm 2's reaction function, i.e. $a_2^S = r_2\left(a_1^S\right)$. Rearranging the FOC we have

$$\frac{dr_2}{da_1} = -\frac{\partial\pi_1/\partial a_1}{\partial\pi_2/\partial a_2}$$

which means that the reaction curve must be tangent to firm 1's isoprofit curve.

3.4 Sequential versus Simultaneous Actions

An interesting question is whether firm 1 will perform more or less of a_1 than it does when it moves simultaneously with firm 2. This depends ultimately on the sign of the second term in (Sequential FOC) as it implies

$$\frac{\partial\pi_1}{\partial a_1}\left(a_1^S, r_2\left(a_1^S\right)\right) = -\frac{\partial\pi_1}{\partial a_2}\left(a_1^S, r_2\left(a_1^S\right)\right)\frac{dr_2\left(a_1^S\right)}{da_1}$$

Recall in the simultaneous Nash solution $\frac{\partial\pi_1}{\partial a_1} = 0$ at a_1^N. Because π_1 is concave, we know $\partial\pi_1/\partial a_1$ is decreasing and so a_1^S will be greater than α_1^N if $\partial\pi_1/\partial a_1$ in the above equation is negative. Conversely a_1^S will be less than α_1^N if $\partial\pi_1/\partial a_1$ in the equation above is positive. On the right-hand side of the equation we can see that the $\partial\pi_1/\partial a_1$ is related to two terms we have already encountered. The first term includes $\partial\pi_1/\partial a_2$ which is related to whether firm 2's action has a positive or negative externality. The second term involves dr_2/da_1 whose sign depends on whether a_1 and a_2 are strategic substitutes or complements. Because there are four different cases we can make a little chart to separate the various cases (work through them to see if you understand).

	Strategic Complement $\frac{\partial^2\pi}{\partial a_1\partial a_2} > 0$	Strategic Substitute $\frac{\partial^2\pi}{\partial a_1\partial a_2} < 0$
Positive Externality $\frac{\partial\pi}{\partial a} > 0$	$a_1^S > a_1^N$ $a_2^S > a_2^N$	$a_1^S < a_1^N$ $a_2^S > a_2^N$
Negative Externality $\frac{\partial\pi}{\partial a} < 0$	$a_1^S < a_1^N$ $a_2^S < a_2^N$	$a_1^S > a_1^N$ $a_2^S < a_2^N$

In the chart we were also able to determine how firm 2's action would be different: when a_1 and a_2 are strategic complements a_2 will be higher if a_1 is higher, while if a_1 and a_2 are strategic substitutes, a_2 will be lower if a_1 is higher (and vice-versa).

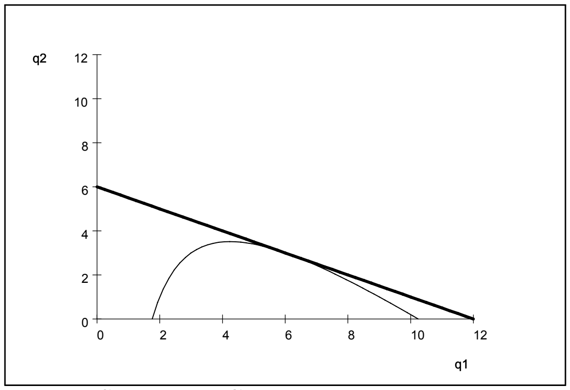
Example 5 ***Cournot Competition*** *The reaction curve for firm 2 is $r_2(q_1) = 6 - q_1/2$. Plugging this into firm 1's profit function gives*

$$\pi_1(q_1, r_2(q_1)) = (12 - q_1 - 6 + q_1/2)\,q_1 = (6 - q_1/2)\,q_1$$

Taking the FOC we get $6 - q_1 = 0 \Rightarrow q_1^S = 6$ and $q_2^S = r_2(6) = 3$.[5] Profits are given by $\pi_1(6,3) = 18$ and $\pi_2(6,3) = 9$ so here firm 1 benefits from a "first mover advantage" as it gets more profits than firm 2. Notice that $q_1^S = 6 > 4 = q_1^N$ and $q_2^S = 3 < 4 = q_2^N$ as q_1 and q_2 are strategic substitutes and exert negative externalities on the other firm, corresponding to the case in the lower right-hand corner of the table above. The tangency of the isoprofit curve of firm 1 and the best response curve of firm 2 mentioned above is shown

[5] It is just a coincidence that the Stackelberg quantity is the same as the monopoly quantity $q_1^S = q^M$.

in the graph below.

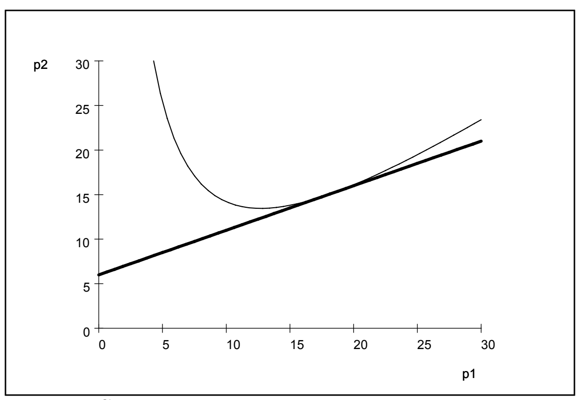


Sequential Cournot Equilibrium

Example 6 ***Hotelling Competition*** *Pluggin in the reaction function of firm 2,* $r_2(p_1) = 6 + p_1/2$ *into firm 1's profit function we get*

$$\begin{aligned} \pi_1(p_1, r_2(p_1)) &= (12 - p_1 + 6 + p_1/2)\, p_1 \\ &= (18 - p_1/2)\, p_1 \end{aligned}$$

Taking FOC this time we get $18 - p_1 = 0 \Rightarrow p_1^S = 18$ *and* $p_2^S = r_2(18) = 15$. *Profits are given by* $\pi_1(18, 15) = 162$ *and* $\pi_2(18, 15) = 225$ *. With Hotelling competition firm 2 gets higher profits as there is a "second mover advantage" unlike the Cournot model. Both firm 1 and firm 2's prices are higher than in the simultaneous Nash equilibrium case as higher prices exert positive externalities on competitors and as prices are strategic complements (the upper right-hand side of the table above). Below is a graph showing the tangency condition - note that profits are higher above the isoprofit curve (with higher* p_2*) than below it.*



Sequential Hotelling Equilibrium