Economics 101A
Section Notes
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More on Consumer Theory: Identities and Slutsky's Equation

1 Identities Relating the UMP and the EMP

The UMP and EMP discussed earlier are mathematically known as "dual" problems. What is a constraint in one is the objective in the other and vice-versa. Furthermore their solutions are very similar, and both are characterized by the same tangency condition, with the only major difference coming from the difference in the constraints: the budget constraint for the UMP and minimum utility constraint for the EMP.

Some useful identities can be given on how their solutions are related. As shown earlier the maximum of utility attained in the UMP is given by the indirect utility function $V(p_x, p_y, I)$. If we substitute the indirect utility function in for the minimum level utility in the EMP, i.e. we set the constraint $U(x,y) \geq V(p_x, p_y, I) = u$, and minimize expenditure $p_x x + p_y y$. The solution of this particular EMP will be identical to the initial UMP. This can be seen since we will be along the same indifference curve as we ended up in the UMP (characterized by $U(x,y) = V(p_x, p_y, I) = u$ - remember since p_x, p_y, and I are given $V(p_x, p_y, I)$ is just a number) and since we will be at a point where

$$MRS(x,y) = \frac{\frac{\partial U}{\partial x}(x,y)}{\frac{\partial U}{\partial y}(x,y)} = \frac{p_x}{p_y}$$

with an amount of expenditure equal to the amount of income I given in the UMP. Therefore $x^c = x^d$ and $y^c = y^d$ if $u = V(p_x, p_y, I)$. Writing out the fact $x^c = x^c(p_x, p_y, u) = x^c(p_x, p_y, V(p_x, p_y, I))$ and $x^d = x^d(p_x, p_y, I)$, and similarly for y, we get the following identities.

$$x^d(p_x, p_y, I) = x^c(p_x, p_y, V(p_x, p_y, I)) \quad \text{(ID1a)}$$

$$y^d(p_x, p_y, I) = y^c(p_x, p_y, V(p_x, p_y, I)) \quad \text{(ID1b)}$$

Furthermore the amount of expenditure needed to attain that level of utility will be equal to the amount of income given in UMP: $E(p_x, p_y, u) = I$, or substituting in for $u = V(p_x, p_y, I)$ we get the identity

$$E(p_x, p_y, V(p_x, p_y, I)) = I \quad \text{(ID2)}$$

Now say that instead of solving the UMP initially we solve the EMP initially. Now say we take the expenditure function from the EMP, $E(p_x, p_y, u)$ and make that the level of income in the budget constraint for the UMP, i.e. we set $p_x x + p_y y \leq E(p_x, p_y, u) = I$ and we maximize $U(x,y)$. Because of the budget constraint the solution will have to be along the same line that defined by the minimum expenditure in the EMP, and characterized by the same tangency condition, with the maximum level of utility attained being equal to the minimum level of utility required in the UMP. Therefore $x^d = x^c$ and $y^d = y^c$ if $I = E(p_x, p_y, u)$ or more formally

$$x^c(p_x, p_y, u) = x^d(p_x, p_y, E(p_x, p_y, u)) \quad \text{(ID3a)}$$

$$y^c(p_x, p_y, u) = y^d(p_x, p_y, E(p_x, p_y, u)) \quad \text{(ID3b)}$$

and since the maximum level of utility attained in the UMP, given by the indirect utility function $V(p_x, p_y, u)$, is given by the minimum level of utility in the EMP, u, we have the last identity[1]

$$V(p_x, p_y, E(p_x, p_y, u)) = u \quad \text{(ID4)}$$

[1] Two less important identities concern the Lagrange multipliers for the UMP (α^d) and the EMP (β^c). Examination of the FOC for these problems reveal that

$$\alpha^d(p_x, p_y, I) = \frac{1}{\beta^c(p_x, p_y, V(p_x, p_y, I))} \text{ and } \beta^c(p_x, p_y, u) = \frac{1}{\alpha^d(p_x, p_y, E(p_x, p_y, u))}$$

These identities can be used to simplify problem solving. For instance, solving the UMP one gets x^d, y^d, α^d and V. Setting $V(p_x, p_y, I) = u$, and solving for I in terms of p_x, p_y, and u yields $I = E(p_x, p_y, u)$ which can then be differentiated to get x^c, y^c and β^c using Shepard's Lemma. Similarly one can proceed from the EMP, and solve the equation $E(p_x, p_y, u) = I$ for u in terms of (p_x, p_y, I) to get $u = V(p_x, p_y, I)$, which can then be used to get x^d, y^d, and α^d, using Roy's Identity. Also one can substitute in $E(p_x, p_y, u)$ for I in x^d to get x^c, or substitute in $V(p_x, p_y, I)$ for u in x^c to get x^d.

Example 1 Continuing with the example with $U(x, y) = x + \log(y)$ (recall $y^d = y^c = p_x/p_y$, $x^d = I/p_x - 1$, $x^c = u - \log p_x + \log p_y$, $V(p_x, p_y, I) = I/p_x - 1 + \log p_x - \log p_y$, $E(p_x, p_y, u) = p_x(u + 1 - \log p_x + \log p_y)$) The case of y is trivial since neither depends on I or u, so

$$y^d(p_x, p_y, I) = \frac{p_x}{p_y} = y^c(p_x, p_y, u)$$

including the cases where $I = E(p_x, p_y, u)$ and $u = V(p_x, p_y, I)$. The case of x simplifies fairly quickly as

$$\begin{aligned}
x^c(p_x, p_y, V(p_x, p_y, I)) &= V(p_x, p_y, I) - \log(p_x) + \log(p_y) \\
&= I/p_x - 1 + \log p_x - \log p_y - \log(p_x) + \log(p_y) \\
&= I/p_x - 1 \\
&= x^d(p_x, p_y, I) \\
x^d(p_x, p_y, E(p_x, p_y, u)) &= \frac{E(p_x, p_y, u)}{p_x} - 1 \\
&= \frac{p_x}{p_x}(u + 1 - \log p_x + \log p_y) - 1 \\
&= u - \log p_x + \log p_y \\
&= x^c(p_x, p_y, u)
\end{aligned}$$

The identities relating the expenditure and indirect utility functions also simplify quickly[2]

$$\begin{aligned}
E(p_x, p_y, V(p_x, p_y, I)) &= p_x(V(p_x, p_y, I) + 1 - \log p_x + \log p_y) \\
&= p_x(I/p_x - 1 + \log p_x - \log p_y + 1 - \log p_x + \log p_y) \\
&= p_x(I/p) \\
&= I \\
V(p_x, p_y, E(p_x, p_y, u)) &= E(p_x, p_y, u)/p_x - 1 + \log p_x - \log p_y \\
&= \frac{p_x}{p_x}(u + 1 - \log p_x + \log p_y) - 1 + \log p_x - \log p_y \\
&= u + 1 - \log p_x + \log p_y - 1 + \log p_x - \log p_y \\
&= u
\end{aligned}$$

2 The Slusky Equation

A very important relating the effect of a price change on uncompensated demands with the effect of a price change in compensated demands as well as an income change can be derived from the identity given in (ID3a) which states that for any given u (and any corresponding level of income $I = E(p_x, p_y, u)$)

$$x^c(p_x, p_y, u) = x^d(p_x, p_y, E(p_x, p_y, u))$$

[2] Also note that the Lagrange multipliers work out simply as for any u or I

$$\alpha^d(p_x, p_y, I) = \frac{1}{p_x} = \frac{1}{\beta^c(p_x, p_y, u)}$$

Differentiating this identity totally with respect to p_x gives

$$\frac{\partial x^c(p_x,p_y,u)}{\partial p_x} = \frac{\partial x^d(p_x,p_y,E(p_x,p_y,u))}{\partial p_x} + \frac{\partial x^d(p_x,p_y,E(p_x,p_y,u))}{\partial I}\frac{\partial E(p_x,p_y,u)}{\partial p_x}$$

Using Shepard's Lemma to substitute in $x^c = \partial E/\partial p_x$ we get

$$\frac{\partial x^c(p_x,p_y,u)}{\partial p_x} = \frac{\partial x^d(p_x,p_y,E(p_x,p_y,u))}{\partial p_x} + \frac{\partial x^d(p_x,p_y,E(p_x,p_y,u))}{\partial I}x^c(p_x,p_y,u)$$

Now solving for $\partial x^d/\partial p_x$ and substituting in (ID3a) we have

$$\frac{\partial x^d(p_x,p_y,E(p_x,p_y,u))}{\partial p_x} = \frac{\partial x^c(p_x,p_y,u)}{\partial p_x} - \frac{\partial x^d(p_x,p_y,E(p_x,p_y,u))}{\partial I}x^d(p_x,p_y,E(p_x,p_y,u))$$

Note that we don't have to have use u as an argument in this equation, as it is completely general, and substitute in any corresponding value of I we like, replacing $E(p_x,p_y,u)$ with I, and u with $V(p_x,p_y,I)$ to get the **Slutsky equation**

$$\frac{\partial x^d(p_x,p_y,I)}{\partial p_x} = \frac{\partial x^c(p_x,p_y,V(p_x,p_y,I))}{\partial p_x} - \frac{\partial x^d(p_x,p_y,I)}{\partial I}x^d(p_x,p_y,I) \qquad \text{(Slutsky)}$$

The first term on the right hand side of the Slutsky equation $\frac{\partial x^c}{\partial p_x}$ is always negative[3] and is commonly known as the **substitution effect**. The second term $-\frac{\partial x^d}{\partial I}x^d$ is known as the **income effect** and is typically, not always, negative, depending on whether $\frac{\partial x^d}{\partial I} > 0$, i.e. whether x is a normal good.

The Slutsky equation can also be expressed in terms of elasticities. First we must define the following: the **price elasticities for uncompensated and compensated demand**

$$e_{x^d,p_x} = \frac{\partial x^d}{\partial p_x}\frac{p_x}{x^d}, \quad e_{x^c,p_x} = \frac{\partial x^c}{\partial p_x}\frac{p_x}{x^c}$$

the **income elasticity of demand**

$$e_{x^d,I} = \frac{\partial x^d}{\partial I}\frac{I}{x^d}$$

and the **share of income spent on** x as

$$s_x = \frac{p_x x^d}{I}$$

Multiplying the Slutsky equation $\frac{\partial x^d}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x^d}{\partial I}x^d$ by p_x/x we get

$$e_{x^d,p_x} = e_{x^c,I} - s_x e_{x^d,I}$$

Example 2 With quasilinear utility we saw $y^d = y^c$ and this can be attributed partly to the fact that there is no income effect $\partial y^d/\partial I = 0$ and so

$$\frac{\partial y^d}{\partial p_y} = \frac{\partial y^c}{\partial p_y} = \frac{\partial}{\partial p_y}\left(\frac{p_x}{p_y}\right) = -\frac{p_x}{p_y^2}$$

This expression is less than zero. For x on the other hand, all marginal income goes to buying it as

$$\frac{\partial x^d}{\partial I} = \frac{1}{p_x}$$

[3] The fact that $\frac{\partial x^c}{\partial p} < 0$ follows from the fact that $E(p_x,p_y,u)$ is a concave function in p_x and so it's second derivative is negative, so $0 > \frac{\partial^2 E}{\partial p_x^2} = \frac{\partial}{\partial p_x}\frac{\partial E}{\partial p_x} = \frac{\partial}{\partial p_x}x^c = \frac{\partial x^c}{\partial p_x}$. For a proof of why $E(p_x,p_y,u)$ is concave consult your notes.

Multiplying by x^d gives the income effect

$$-\frac{\partial x^d}{\partial I}x^d = -\frac{1}{p_x}\left(\frac{I}{p_x}-1\right) = -\frac{I}{p_x^2}+\frac{1}{p_x} = -\frac{I-p_x}{p_x^2}$$

which is negative since $I > p_x$ (assuming $x^d > 0$, otherwise if $p_x \geq I$, $x^d = 0$ and the income effect is zero). The substitution effect is given by

$$\frac{\partial x^c}{\partial p_x} = -\frac{1}{p_x}$$

Which is obviously negative. The Slutsky's equation holds as

$$\frac{\partial x^d}{\partial p_x} = -\frac{I}{p_x^2} = -\frac{1}{p_x} + \left(-\frac{I}{p_x^2}+\frac{1}{p_x}\right) = \frac{\partial x^c}{\partial p_x} - \frac{\partial x^d}{\partial I}x^d$$

The elasticities are given by

$$\begin{aligned} e_{x^d,p_x} &= -\frac{I}{p_x^2}\frac{p_x}{I/p_x-1} = -\frac{I}{I-p_x} & e_{y^d,p_y} &= -\frac{p_x}{p_y^2}\frac{p_y}{p_x/p_y} = -1 \\ e_{x^c,I} &= -\frac{1}{p_x}\frac{p_x}{I/p_x-1} = -\frac{p_x}{I-p_x} & e_{y^c,p_y} &= -\frac{p_x}{p_y^2}\frac{p_y}{p_x/p_y} = -1 \\ e_{x^d,I} &= \frac{1}{p_x}\frac{I}{I/p_x-1} = \frac{I}{I-p_x} & e_{x^d,I} &= 0 \end{aligned}$$

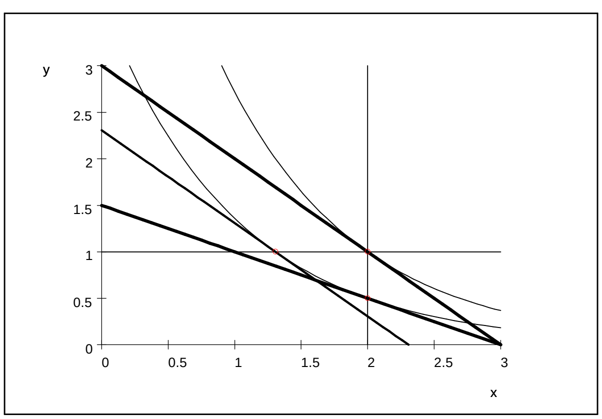
Now the share of income spent on each good is given by

$$s_x = \frac{p_x\left(I/p_x-1\right)}{I} = \frac{I-p_x}{I} \qquad s_y = \frac{p_y\left(p_x/p_y\right)}{I} = \frac{p_x}{I}$$

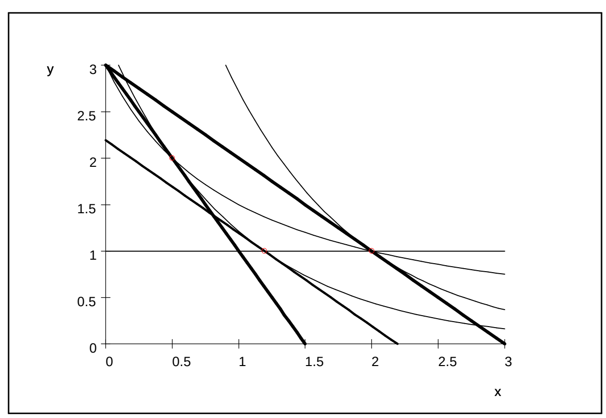
The Slutsky equation for y is trivial and for x is easily checked

$$\begin{aligned} e_{x^c,I} - s_x e_{x^d,I} &= -\frac{p_x}{I-p_x} - \left(\frac{I-p_x}{I}\right)\frac{I}{I-p_x} \\ &= -\frac{p_x}{I-p_x} - 1 \\ &= -\frac{I}{I-p_x} \\ &= e_{x^d,p_x} \end{aligned}$$

Try working out the meaning and exact expressions for all of the quantities and curves seen in these graphs which use $U(x,y) = x + \log y$.



$p_x = 1, I = 3, p_y^1 = 1, p_y^2 = 2$



$p_y = 1, I = 3, p_x^1 = 1, p_x^2 = 2$