

The Neoclassical Firm

1 Setup of the Neoclassical Firm

- One output q , with price p and two inputs, labor (or "employment") E that must be paid wage w and capital K , which must be paid a rental rate r .
- Because of perfect competition, the firm takes prices p, w and r as given.
- Firms maximize profits $\pi = pq - wE - rK$ by choosing q, E and K .
- Firms are subject to the production constraint $q \leq f(E, K)$ where f is the production function for q .
- The example we will use in section is production function $f(E, K) = E^{1/2} + K^{1/2}$

2 The Firm's Profit Maximization Problem

The firm's maximization problem is given by

$$\max_{q, E, K} pq - wE - rK \quad \text{s.t. } f(E, K) \geq q \quad (\text{Full PMP})$$

To solve the firm problem we make use of the Lagrangean

$$\mathcal{L}(q, E, K, \lambda) = pq - wE - rK + \lambda [f(E, K) - q] \quad (\text{Firm Lagrangean})$$

The four first order conditions are given by

$$\frac{\partial \mathcal{L}}{\partial q} = p - \lambda = 0 \quad (\text{q FOC})$$

$$\frac{\partial \mathcal{L}}{\partial E} = w - \lambda \frac{\partial f}{\partial E} = 0 \quad (\text{E FOC})$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{\partial f}{\partial K} = 0 \quad (\text{K FOC})$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = f(E, K) - q = 0 \quad (\text{Production Constraint})$$

The first equation (q FOC) implies that $p = \lambda$, so that the Lagrange multiplier on the production constraint, the "shadow price" of output, should equal the market price of output. As we will see later this is actually a statement of the familiar principle that a competitive firm should keep producing a good until its marginal cost rises to equal its price, i.e. $p = MC$.

Using $p = \lambda$, the next two FOC can be rearranged to give

$$w = p \frac{\partial f}{\partial E} = p \cdot MP_E = VMP_E \quad (\text{VMP E condition})$$

$$r = p \frac{\partial f}{\partial K} = p \cdot MP_K = VMP_K \quad (\text{VMP K condition})$$

which states that that the firm should keep hiring labor until its **marginal revenue product** VMP_E (equal to p times the **marginal product** MP_E) falls to level of the wage w , and similarly should keep hiring capital until its marginal revenue product, VMP_K , fall to the rental rate r . The final equation just restates the production equation.

The four FOC specify a system of four equations in four unknowns (q, E, K, λ) and can therefore be solved to yield **product supply** $q^S(p, w, r)$, and (unconditional) **labor and capital demand** $E^D(p, w, r)$, $K^D(p, w, r)$ and λ which is the marginal cost of q . Finally, one can substitute all of the solutions into the objective function to get the **profit function**

$$\Pi(p, w, r) = pq^S(p, w, r) - wE^D(p, w, r) - rK^D(p, w, r) \quad (\text{Profit Fct})$$

3 Sub-Problems

The firm's profit maximization problem can be seen as a combination of two problems: (1) a cost minimization problem, and (2) a simple profit maximization problem with a cost function.

3.1 The Cost Minimization Problem

Suppose the firm must choose a given level of output q_0 , but can adjust its level of labor and capital in an optimal way, then it would essentially just solve

$$\max_{E, K} -wE - rK \quad \text{s.t. } f(E, K) \geq q_0 \quad (1)$$

As minimizing the negative of something is equivalent to maximizing it, this is equivalent to the **cost minimization problem**

$$\min_{E, K} wE + rK \quad \text{s.t. } f(E, K) \geq q_0 \quad (\text{Cost Min})$$

The corresponding Lagrangian function

$$\mathcal{L}(E, K, \lambda) = wE + rK + \lambda[q_0 - f(E, K)]$$

leads to the same FOC as the profit maximization problem without (q FOC) as output q_0 is not a control variable and there is no dependence on output price p . Combining (E FOC) and (K FOC) we get the tangency condition which characterizes the cost minimization problem

$$\frac{w}{r} = \frac{\partial f / \partial E}{\partial f / \partial K} = MRTS_{EK} \quad (\text{Tangency})$$

where $MRTS_{EK}$ is the **marginal rate of technical substitution** between labor and capital, i.e. how many units of capital must be added to replace one fewer unit of labor, keeping output constant. This is the conventional problem of finding an **isocost** curve (with slope $-w/r$) which is tangent to a required **isoquant** curve (with slope $-MRTS_{EK}$). Using the production constraint and the tangency condition one can solve for **conditional labor and capital demands** $E^C(q_0, w, r)$, $K^C(q_0, w, r)$ which depend on q_0 rather than p . Substituting these conditional demands into the objective of (Cost Min) leads to the **cost function**

$$C(q_0, w, r) = wE^C(q_0, w, r) + rK^C(q_0, w, r) \quad (\text{Cost Fct})$$

Finding the multiplier $\lambda(q_0, w, r)$ is no longer trivial as well as it no longer needs to equal p . It is really in this context that it is possible to understand how λ is the marginal cost since it is the shadow price of producing an extra unit of q in terms of cost, i.e. the cost on the margin. This is made clear using the envelope theorem

$$\frac{\partial C}{\partial q_0} = \frac{\partial \mathcal{L}}{\partial q_0} = \lambda \quad (\text{MC})$$

It is in the context of the cost minimization problem that the **elasticity of substitution**, call it " σ_{Subs} ", is defined

$$\sigma_{Subs} = - \frac{\partial (E^C / K^C)}{\partial (w/r)} \frac{(w/r)}{(E^C / K^C)} \quad (\text{Elast. of Substitution})$$

The higher the elasticity of substitution the more a firm will substitute towards capital and away from labor (for a given level of output) when the cost of labor rises or the price of capital falls.

3.2 Simple Profit Maximization Problem with a Cost Function

Given the cost function, we can maximize the firm's profit by choosing output q optimally by solving the following problem

$$\max_q pq - C(q, w, r) \quad (\text{Simple PMP})$$

This is an unconstrained maximization problem in one variable with one FOC, namely

$$p - \frac{\partial C}{\partial q} = 0 \quad (p = MC)$$

or rearranging $p = \partial C / \partial q = MC$, i.e. that price equals marginal cost. Using the fact that $\partial C / \partial q = \lambda$, this condition is equivalent to (q FOC) in the full profit maximization problem we first considered. Solving this equation for q leads to output supply function $q^S(p, w, r)$. As the simple profit maximization problem assumes that firms are choosing inputs so as to minimize costs unconditional factor demands can be found merely by substituting the output supply function into the conditional factor demands

$$\begin{aligned} E^D(p, w, r) &= E^C(q^S(p, w, r), w, r) \\ E^D(p, w, r) &= E^C(q^S(p, w, r), w, r) \end{aligned}$$

Therefore the full profit maximization problem can be considered as a two-step problem: (1) A cost minimization problem characterized by the (Tangency) and (Production Constraint) conditions, and (2) a simple profit maximization problem with a cost function characterized by the ($p = MC$) condition.

3.3 The Short Run

The problems considered above are sometimes considered to be in the "long run" as capital can be adjusted. The short run version of these problems are the same except that capital supply is fixed at a level K_0 . In the profit maximization problem, not being able to choose this level means that (K FOC) should be ignored as K is not a choice variable, which eliminates r from the analysis as capital is a sunk cost. Short run product supply, labor demand, and profits depend additionally on K_0 but no longer on r , i.e., we will find functions $q^S(p, w, K_0)$, $E^D(p, w, K_0)$, and $\Pi(p, w, K_0)$. In this case the elasticity of labor demand tends to be more inelastic as the firm cannot substitute towards capital if the wage rate increases.

In the cost minimization problem, the inability to choose K_0 leaves only one free choice variable E which must be used to satisfy the production constraint $q_0 = F(E, K_0)$, making the condition (E FOC) irrelevant as there are no other margins which can be adjusted as both capital and output are fixed. Conditional labor demand is found simply by inverting the production function in the E argument - i.e. solving $q_0 = F(E^C(q_0, K_0), K_0)$ for $E^C(q_0, K_0)$ - and the cost function is given simply by $C(q_0, w, K_0) = wE^C(q_0, K_0)$. The simple profit maximization problem is no different in the short run than in the conventional long run problem.

Note that the long-run problems can be solved from their short-run counterparts by maximizing $\Pi(p, w, K_0)$ or minimizing $C(q_0, w, K_0)$ over K_0 .

4 The Multi-Plant Firm

Suppose that a firm has N plants, each with concave production functions $f_i(E_i, K_i)$, and that each plant minimizes costs over its inputs, leading to cost functions $C_i(q_i, w, r)$. Suppressing the notation w and r we can write the cost functions as $C_i(q_i)$. Now say the firm wishes to produce a total quantity q at all of its plants, minimizing its total costs. In other words it wishes to solve the following problem.

$$\min_{q_1, \dots, q_N} \sum_{i=1}^N C_i(q_i) \quad s.t. \quad \sum_{i=1}^N q_i \geq q \quad (\text{Multi-plant Problem})$$

Writing out the Lagrangean we have

$$\mathfrak{L}(q_1, \dots, q_N, \lambda) = \sum_{i=1}^N C_i(q_i) + \lambda \left(q - \sum_{i=1}^N q_i \right)$$

The first order conditions are given by

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q_i} &= \frac{dC_i(q_i^*)}{dq_i} - \lambda^* = 0 \quad i = 1, \dots, N \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= q - \sum_{i=1}^N q_i^* = 0\end{aligned}$$

The first N equations imply that $MC_i(q_i^*) = \lambda^*$ for all plants¹, which implies that the marginal costs of each plant is the same, i.e. $MC_i(q_i^*) = MC_j(q_j^*)$. Because marginal costs are increasing in q_i and q_j then if $MC_i(q_i) > MC_j(q_j)$ the firm could reduce costs by lowering q_i and increasing q_j .

As we have $N+1$ equations in $N+1$ unknowns ($q_1^*, \dots, q_N^*, \lambda^*$), we can solve for the plant-level outputs $q_i^*(q)$ and the Lagrange multiplier $\lambda^*(q)$. The firm-level cost function is given by $C(q) = \sum_{i=1}^N C_i(q_i^*(q))$. Note that by the envelope theorem, the firm's marginal cost is

$$\frac{dC(q)}{dq} = \frac{\partial \mathcal{L}}{\partial q} = \lambda^*(q) \quad (\text{Firm's MC})$$

which is the same as each plant-level marginal costs, i.e. $MC(q) = \lambda^*(q) = MC_i(q_i^*(q))$ for all i .

Graphically the plant-level marginal cost curve is the horizontal summation of the individual-plant marginal cost curves where q is on the x-axis and λ^* is the quantity on the y-axis (recall this equals p when firms maximize profits). Analytically this can be seen by inverting the FOC $MC_i(q_i^*) = \lambda^*$ to yield $q_i^* = MC_i^{-1}(\lambda^*)$ and the firm's MC condition $MC(q) = \lambda^*$ to yield $q = MC^{-1}(\lambda^*)$. Using the production requirement

$$\sum_{i=1}^N MC_i^{-1}(\lambda^*) = \sum_{i=1}^N q_i^* = q = MC^{-1}(\lambda^*)$$

If by mistake you were to add up the curves vertically you would not get the firm's marginal cost, but N times it, i.e. $\sum_{i=1}^N MC_i(q_i^*) = N\lambda^* = N \cdot MC(q) \neq MC(q)$.²

As a last remark, note that the equalization of marginal costs are also implies the equalization of marginal products of factors across plants, e.g. as $w = \lambda^* \frac{\partial f_i}{\partial E_i}$ for all i , then $\frac{\partial f_i}{\partial E_i} = \frac{w}{\lambda^*} = \frac{\partial f_j}{\partial E_j}$ for all i and j , i.e. $MP_{E_i} = MP_{E_j}$

Example 1 In section we used the example $C_1(q_1) = q_1$ and $C_2(q_2) = \frac{1}{2}(q_2)^2$.

¹Taking into account the non-negativity constraint that each $q_i \geq 0$, then each FOC implies that $MC_i(q_i^*) \geq \lambda^*$ and is guaranteed to hold with equality only if $q_i^* > 0$. If $MC_i(0) > \lambda^*(q)$ then $q_i^*(q) = 0$, i.e. the firm produces zero at plant i and does production at other plants.

²Note that if you differentiate the condition $C(q) = \sum_{i=1}^N C_i(q_i^*(q))$ you get

$$MC(q) = \sum_{i=1}^N MC_i(q_i^*(q)) \frac{dq_i^*}{dq} = \lambda^* \sum_{i=1}^N \frac{dq_i^*}{dq} = \lambda^*$$

as $\sum_{i=1}^N \frac{dq_i^*}{dq} = 1$ by the production constraint.