Economics 250c
Dynamic Discrete Choice, continued

This lecture will continue the presentation of dynamic discrete choice problems with extreme value errors. We will discuss:

1. Ebenstein's model of sex selection and fertility (a very simple DDC model)
2. a "more structural" variant of Ebenstein
3. Infinite horizon/stationary models

Some references:

John Rust. "Estimation of Dynamic Structural Models, Problems and Prospects: Discrete Decision Processes". In C. Sims, editor. Advances in Econometrics Sixth World Congress, Volume II. Cambridge University Press, 1994.

John Rust. "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher." Econometrica 55 (Sept. 1987): 999-1033.

Avi Ebenstein. "Estimating a Dynamic Model of Sex Selection in China".

Prologue

Often we want to scale the errors in a choice model. If the structural error term is e, a convenient assumption is $e/\tau = \epsilon \sim EV1$. If we have a random choice setup with $u_j = v_j + e_j$, then we can write $u_j/\tau = v_j/\tau + e_j/\tau = v_j/\tau + \epsilon_j$, where $\epsilon_j \sim EV1$. Thus

$$P(d_j = 1|v_1, ..v_J) = \frac{\exp(v_j/\tau)}{\Sigma_k \exp(v_k/\tau)}$$

and

$$E(e_j|d_j = 1) = \tau(\gamma - \log p_j)$$

and

$$E \max(u_1, ...u_J) = \tau(\gamma + \log(\Sigma_k \exp(v_k/\tau)))$$

1.Ebenstein's model

Ebenstein presents a simple "dynamic" model of fertility and sex selection, designed to explain how the imposition of fines for additional children increases the incentive for parents to use sex-selective abortion. This is an interesting variant of the "quality-quantity" tradeoff hypothesis (Becker and Lewis, JPE, 1973). In principle the model can be used to anwer policy questions, such as "what would happen to the number and gender of births if the fines were changed?" We will discuss a stripped down version of his model to illustrate the basic mechanics of a finite DDC model. The key assumptions are:

-parents only care about having a boy.
-utility is additive in income and the utility-equivalent of completed fertility
-parents face a fine of F_2 for the second child and F_3 for the third
-parents costlessly observe the sex of an unborn child
-for an outlay of A, an unborn female can be converted to a boy birth (we will show how to replace this with a more structural assumption in section 2).
-families have at most 3 kids.
-the probability of a boy is 0.51 at every conception.

The state variables in this model are the number and gender distribution of kids. We will use the notation "GB" to mean a family that has 1 girl (born first) and 1 boy (born second). The notation "GGg" means that the family has 2 girl children and an unborn girl. The assumptions above mean that once a family has a boy, they stop having additional children. Thus, the only possible fertility outcomes are: B, G, GB, GG, GGB, GGG. The non-random components of the money-equivalent utilities assigned to final outcomes B, GB, and GGB are all the same, and equal to θ. The non-random components of the utilities assigned to the final outcomes G, GG, and GGG are all 0. As discussed in Lecture 6, however, we will follow Rust and add a re-scaled EV-1 error at each stage of the process. In part 2 we will illustrate via an example how this simplifies the DDC.

The decision tree for this model is shown in Figure 1. There are five decision nodes. In order, these are:

node 1: whether to abort g and convert to B, if first conception is female
node 2: whether to continue having another child if state is G
node 3: whether to abort Gg and convert to GB, if 2^{nd} conception is female
node 4:whether to continue having another child if state is GG
node 5: whether to abort GGg and convert to GGB, if 3^{rd} conception is female

Working backward, at node 5 the state is GGg and the choices are:
abort (cost A): $V_1^5 = \theta - A - F_3 + \epsilon_1^5$
don't abort: $V_0^5 = -F_3 + \epsilon_0^5$,
where $\epsilon/\tau \sim EV1$. Then

$$
\begin{aligned}
P(GGB|GGg) &= \frac{\exp((\theta - A - F_3)/\tau)}{\exp(-F_3/\tau) + \exp((\theta - A - F_3)/\tau)} \\
&= \frac{\exp((\theta - A)/\tau)}{1 + \exp((\theta - A)/\tau)} \quad , \quad \text{and} \\
E\max(GGg) &= \tau(\gamma + \log(\exp(-F_3/\tau) + \exp((\theta - A - F_3)/\tau) \\
&= -F_3 + \tau(\gamma + \log(\ 1 + \exp(\theta - A)/\tau \ \ )) \\
&= -F_3 + k^5(\theta - A), \ \text{ where } k^5(\cdot) \text{ is increasing.}
\end{aligned}
$$

Note that when θ is much bigger than A, the family almost always chooses to abort, and $E\max(GGg) \to -F_3 + (\theta - A)$. The function $k^5(\theta - A)$ is the "option value" of GGg.

Now go back to node 4. Here the state is GG and the choices are:
stop: $V_{stop}^4 = \epsilon_{stop}^4$,
continue: $V_{cont}^4 =$ .49 chance at GGg, payoff = Emax(GGg)
.51 chance at GGB, payoff = $\theta - F_3$.

Thus

$$
\begin{aligned}
V_{cont}^4 &= .49(-F_3 + k^5) + .51(\theta - F_3) + \epsilon_{cont}^4 \\
&= .51\theta + 49k^5 - F_3 + \epsilon_{cont}^4
\end{aligned}
$$

and so the probability of continuing fertility, having reached GG is

$$
P(3rd|GG) = \frac{\exp((.51\theta + 49k^5 - F_3)/\tau)}{1 + \exp((.51\theta + 49k^5 - F_3)/\tau)}
$$

Note that this is increasing in θ and the option value of GGg, and decreasing in F_3. Moreover (in slightly incorrect notation) the expected value from GG forward is

$$E\max(GG) = \tau(\gamma + \log(1 + \exp(.51\theta + 49k^5 - F_3))).$$

Now go back to node 3, where the state is Gg. As at node 5, the choices are

abort (cost A): $V_1^3 = \theta - A - F_2 + \epsilon_1^3$

don't abort: $V_0^5 = E\max(GG) - F_2 + \epsilon_0^3$.

Thus,

$$P(GB|Gg) = \frac{\exp((\theta - A - F_2)/\tau)}{\exp((E\max(GG) - F_2)/\tau) + \exp((\theta - A - F_2)/\tau)},$$

and

$$\begin{aligned} E\max(Gg) &= \tau(\gamma + \log(\exp((E\max(GG) - F_2)/\tau) + \exp((\theta - A - F_2)/\tau) \\ &= -F_2 + \tau(\gamma + \log(\ \exp(E\max(GG)/\tau) + \exp(\theta - A)/\tau\ )) \\ &= -F_3 + k^3(\theta - A, E\max(GG)), \end{aligned}$$

where again the function $k^3(\cdot, \cdot)$ represents the option value of Gg.

At node 2 the state is G and the choice is whether to continue to have a second conception or not. Following the same analysis as at node 4, the choices are

stop: $V_{stop}^2 = \epsilon_{stop}^2$,

continue: $V_{cont}^2 =$ .49 chance at Gg, payoff = Emax(Gg)

.51 chance at GB, payoff = $\theta - F_2$.

Thus

$$\begin{aligned} V_{cont}^2 &= .49(-F_2 + k^3) + .51(\theta - F_2) + \epsilon_{cont}^2 \\ &= .51\theta + 49k^3 - F_2 + \epsilon_{cont}^2 \end{aligned}$$

and the probability of continuing fertility, having reached G is

$$P(2rd|G) = \frac{\exp((.51\theta + 49k^3 - F_2)/\tau)}{1 + \exp((.51\theta + 49k^3 - F_2)/\tau)},$$

while the expected value from G forward is

$$E\max(G) = \tau(\gamma + \log(1 + \exp(.51\theta + 49k^3 - F_2))).$$

Finally, node 1 is analogous to nodes 3 and 5. The state is g and the choices are:

abort (cost A): $V_1^1 = \theta - A + \epsilon_1^1$

don't abort: $V_0^1 = E\max(G) + \epsilon_0^1$.

Therefore

$$\begin{aligned} P(B|g) &= \frac{\exp((\theta - A)/\tau)}{\exp((E\max(G))/\tau) + \exp((\theta - A)/\tau)} \quad \text{and} \\ E\max(g) &= \tau(\gamma + \log(\exp((E\max(G))/\tau) + \exp((\theta - A)/\tau) \\ &= k^1(\theta - A, E\max(G)), \end{aligned}$$

where $k^1(\cdot,\cdot)$ represents the option value of g.

Note that as F_2 and F_3 increase, $E\max(G)$ and $E\max(GG)$ decrease, and the family is more likely to exercise selective abortion on the first or second birth. If $F_2 < F_3$ (as is true in the data) families are more likely to exercise selective abortion for the second parity, both because the fine is higher for third "try," and because the continuation value after G is larger than the continuation value after GG.

With these equations we have derived 5 key probabilities, as functions of $(\theta, A, F_2, F_3, \tau)$:

$$\begin{aligned}
P(GGB|GGg) &= \pi_3 \\
P(GB|Gg) &= \pi_2 \\
P(B|g) &= \pi_1 \\
P(stop|GG) &= 1 - P(3rd|GG) = \rho_2 \\
P(stop|G) &= 1 - P(2nd|G) = \rho_1.
\end{aligned}$$

Using these, it is possible to write down the probability for each completed fertility outcome. These are

$$\begin{aligned}
P(B) &= .51 + .49\pi_1 \\
P(G) &= .49(1-\pi_1)\rho_1 \\
P(GB) &= .49(1-\pi_1)(1-\rho_1)(.51+.49\pi_2) \\
P(GG) &= .49(1-\pi_1)(1-\rho_1)(.49)(1-\pi_2)\rho_2 \\
P(GGB) &= .49(1-\pi_1)(1-\rho_1)(.49)(1-\pi_2)(1-\rho_2)(.51+.49\pi_3) \\
P(GGG) &= .49(1-\pi_1)(1-\rho_1)(.49)(1-\pi_2)(1-\rho_2)(.49)(1-\pi_3).
\end{aligned}$$

Note that by using a dynamic model, we can write everything in terms of the conditional probabilities (and the factors .49 and .51). Given $(\theta, A, F_2, F_3, \tau)$ it is therefore possible to assign a likelihood for an observed fertility outcome. A very useful feature of the additive EV1 setup is that all the conditional probabilities are strictly between 0 and 1. There is never a case where the parameters can't map to a probability for the observed outcome.

Implementation

Ebenstein has data on the fines for different sizes of family (which vary by region). Also, in some regions F_2 is 0 (or small) if the first child is a girl. (This is the so-called $1\frac{1}{2}$ child policy). He also has data on the education (Ed), farm/non-farm status (Farm) of each family, whether they live in a high-rise (Hirise) and an estimate of the travel time to the nearest fertility clinic (Clinic). He assumes that

$$\begin{aligned}
\theta &= \beta_0 + \beta_1 Ed + \beta_2 Farm + \beta_3 Hirise \\
A &= \alpha_0 + \alpha_1 Clinic
\end{aligned}$$

The model gives a likelihood for the observed completed fertility f of a couple, $p(f|F_2, F_3, Ed, Farm, Hirise, Clinic)$ The max. likelihood estimates suggest that farm families and those with less education have much stronger preferences for boys. He also finds a positive estimate α_1. (The latest version of E's paper allows for the family to have final payoffs that depend flexibly on the number and sex composition of the kids).

Note that in principle it would be possible to add a random error ϕ to the equation for θ, with some distribution $G(\phi)$ (for example, a point mass distribution with 2,3... points of support). Then the likelihood would be maximized over the structural parameters $(\beta_0, \beta_1, \beta_2, \beta_3, \alpha_0, \alpha_1)$ and the parameters of G.

2. A "more structural" variant of Ebenstein

It is interesting to look at a slightly different version of Ebenstein's model that puts more structure on the sex-selection process. In particular, suppose the decision process after conception is as follows:

- -decide whether to check sex (i.e., have an ultrasound) at cost c
  - -if check reveals "b", carry to term and receive payoff $\theta - F - c + \epsilon^b_{check}$
  - -if check reveals "g", abort, receive payoff $-A - c + \epsilon^g_{check}$ and return to pre-conception node
- -if no check is made
  - -with probability .51, outcome is B, payoff is $\theta - F + \epsilon^b_{nocheck}$
  - -with probability .49, outcome is G, payoff is $-F + \epsilon^g_{nocheck}$

For concreteness, let's consider the last node in a 3-kid model. The state is GG and the value V_2 represents the Emax of going forward (the subscript 2 stands for "2 girls"). Figure 2 shows the payoffs. Note that the figure includes two "new" error terms, ε_{check} and $\varepsilon_{nocheck}$ that are potentially added to the Emax's associated with "check" and "no check" decisions. In the basic Rust setup, there is always a unique additive EV-1 error associated with each branch of any decision node.

For the check decision, the expected payoff is

$$V_{check} = .51(\theta - c - F) + .49(V_2 - c - A)$$

The presence of V_2 reflects the "recursive" structure of this set-up: if the parents check and decide to abort, they return to the state GG. For the no-check decision the expected payoff is

$$V_{nocheck} = .51(\theta - F) + .49(-F) = .51\theta - F.$$

Lets solve first, ignoring ε_{check} and $\varepsilon_{nocheck}$. If "check" is the optimal choice we have:

$$\begin{aligned} V_2 &= .51(\theta - c - F) + .49(V_2 - c - A) \\ \Rightarrow V_2 &= \theta - F - \frac{c}{.51} - \frac{.49}{.51}A. \end{aligned}$$

This expression can be interpreted as follows. Assuming sex-selection is optimal, V_2 is the sum of $\theta - F$, minus the cost of the expected number of ultrasounds (approximately 2) minus the cost of the expected number of abortions (approximately 1). If "no check" is optimal,

$$V_2 = .51\theta - F.$$

The family follows a deterministic policy, choosing to check if

$$\begin{aligned} \theta - F - \frac{c}{.51} - \frac{.49}{.51}A &> .51\theta - F \\ or\ \ \theta &> \frac{c}{.51 \times .49} + \frac{1}{.51}A. \end{aligned}$$

Now lets do the calculation, allowing for ε_{check} and $\varepsilon_{nocheck}$. Assuming these are EV-1,

$$\begin{aligned}
V_2 &= E\max(V_{check}+\varepsilon_{check}, V_{nocheck}+\varepsilon_{nocheck}), \\
V_2 &= \gamma + \log\left[\exp(.51\theta + .49V_2 - c - .51F - .49A) + \exp(.51\theta - F)\right] \\
(*) \qquad V_2 &= \gamma + .51\theta + \log\left[\exp(.49V_2 - c - .51F - .49A) + \exp(-F)\right]
\end{aligned}$$

Also,

$$P(check) = \frac{\exp(.49V_2 - c - .51F - .49A)}{\exp(.49V_2 - c - .51F - .49A) + \exp(-F)}.$$

The solution to equation (*) is a simple fixed point calculation in V_2. The l.h.s. has slope 1, the right hand side has slope= $.49P(check)$, which is always less than 1. So there will always be at most 1 solution. Figure 3 graphs the two sides of (*) for 2 choices of the parameters (θ, F, c, A).

Note that in general

$$\begin{aligned}
\log(a+b) &= \log(a) - \log(\frac{a}{a+b}), \\
\Rightarrow \log(e^a + e^b) &= a - \log(\frac{e^a}{e^a+e^b}).
\end{aligned}$$

Using this, we have that

$$\log\left[\exp(.49V_2 - c - .51F - .49A) + \exp(-F)\right] = .49V_2 - c - .51F - .49A - \log(P(check)).$$

So for parameters such that $P(check) \approx 1$,

$$V_2 \approx \gamma + .51\theta + .49V_2 - c - .51F - .49A$$

giving the same expression as we had in the deterministic case when "check" is optimal. Likewise if $P(check) \approx 0$

$$V_2 \approx \gamma + .51\theta - F$$

which is the same as in the deterministic case when "no check" is optimal. The presence of ε_{check} and $\varepsilon_{nocheck}$ "smooths" out the shape of the max function. Compare

$$\max(a,b) \text{ vs. } E\max(a,b) = \log(\exp(a) + \exp(b)).$$

Note that to solve the entire model with this "new" way of looking at the sex selection choice we'd have to perform a fixed point exercise for the choices at nodes 5, 3, and 1. After we've solved for V_2, we can work backward and solve the fixed points at node 3 (for V_1) and at node 1 (for V_0). Rust's bus engine paper can be seen as a version of this problem with many "recusive" nodes (one for each of the intervals of engine mileage that he includes in the state space.

3. Infinite horizon/stationary models

Rust (1987) setup.

Consider an infinite horizon discrete time model. The state in period t is (x_t, ϵ_t). The control variable is a vector $d_t \in D(x_t)$, a discrete set. Flow utility in period t is

$$u(x_t, d_t) + \epsilon(d_t).$$

For example, if $D(x_t) = \{1, 2, ...J\}$ the flow utilities are

$$u(x_t, 1) + \epsilon_1,$$
$$u(x_t, 2) + \epsilon_2, ...$$
$$u(x_t, J) + \epsilon_J.$$

Intertemporal utility is discounted at rate $\beta < 1$. The tansition equation is

$$p(x_{t+1}, \epsilon_{t+1}|x_t, \epsilon_t, d_t) = q(\epsilon_{t+1}|x_{t+1})\pi(x_{t+1}|x_t, d_t).$$

This rules out serial correlation in the $\epsilon_t' s$, and any feedback from ϵ_t to x_{t+1}. The $\epsilon_t' s$ are assumed to have support over the entire real line. For this class of problems the Bellman equation for a stationary solution is

$$\begin{aligned}(*) \qquad V(x, \epsilon) &= \max_{d \in D(x)} u(x, d) + \epsilon(d) + \beta \int_{x'} \int_{\epsilon'} V(x', \epsilon') q(\epsilon'|x')\pi(x'|x, d) \\ &= \max_{d \in D(x)} u(x, d) + \epsilon(d) + \beta \int E_{\epsilon'} V(x', \epsilon')\pi(x'|x, d)\end{aligned}$$

Now define

$$\overline{V}(x) = E_\epsilon V(x, \epsilon).$$

Using $(*)$

$$(**) \qquad \overline{V}(x) = E_\epsilon \left[\max_{d \in D(x)} u(x, d) + \epsilon(d) + \beta \int_{x'} \overline{V}(x')\pi(x'|x, d) \right].$$

Note that $\overline{V}(x)$ is the $E\max$ function for a one-period choice problem in which the payoff to choice d is

$$v(x, d) = u(x, d) + \beta \int_{x'} \overline{V}(x')\pi(x'|x, d).$$

That is:

$$\overline{V}(x) = E_\epsilon \left[\max_{d \in D(x)} v(x, d) + \epsilon(d) \right].$$

Moreover, assuming that agents see the vector $\epsilon(d)$ in period t when they make their choice, but we as econometricians do not, the probability of choice d in period t is just

$$P\left(v(x, d_j) + \epsilon(j) \geq v(x, d_k) + \epsilon(k) \text{ for all } k \neq j\right).$$

Equation $(**)$ is a contraction mapping, so it has a fixed point which is the stationary integrated value function. With a finite state space, the value function is just a list of numbers, of length equal to the dimension of the state space, so the finding a solution to the contraction mapping is feasible. In the case of EV-1 errors

$$E_\epsilon \left[\max_{d \in D(x)} v(x, d) + \epsilon(d) \right] = \log \left[\sum_d \exp(v(x, d)) \right]$$

so equation $(**)$ becomes

$$\overline{V}(x) = \log \left[\sum_d \left(\exp(u(x, d) + \beta \int_{x'} \overline{V}(x')\pi(x'|x, d) \right) \right]$$

and the probability of the j^{th} choice is

$$P(d_{jt} = 1|x_t) = \frac{\exp(v(x_t, d_{jt}))}{\sum_k \exp(v(x_t, d_{kt}))}.$$

For given parameters, and a given $\pi(x'|x, d)$ we can find the solution function $\overline{V}(x)$ at each value of the state space, and derive an expression for $P(d_t|x_t)$. The method is to posit a guess $\overline{V}^0(x)$ then apply sequentially

$$\overline{V}^{k+1}(x) = \log\left[\sum_d \left(\exp(u(x, d) + \beta \int_{x'} \overline{V}^k(x')\pi(x'|x, d)\right)\right]$$

until $\overline{V}^{k+1}(x) \approx \overline{V}^k(x)$.