

Economics 101A
(Lecture 3, revised)

Stefano DellaVigna

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Outline

1. Envelope Theorem
2. Convexity and concavity
3. Constrained Maximization

1 Envelope Theorem

- You now know how x_1^* varies if p_1 varies.
- How does the function h vary at the optimum as p_1 varies?
- Differentiate $h(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)$ with respect to p_1 :

$$\begin{aligned} & \frac{dh(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)}{dp_1} \\ = & \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_1} * \frac{\partial x_1^*(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \\ & + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_2} * \frac{\partial x_2^*(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \\ & + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \end{aligned}$$

- Can we say something about the first two terms?
They are zero!

- **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(\mathbf{x}; \mathbf{p})$ with respect to x . Consider then the function f at the optimum, that is, $f(\mathbf{x}^*(\mathbf{p}), \mathbf{p})$. The total differential of this function with respect to p_i equals the partial derivative with respect to p_i :

$$\frac{df(\mathbf{x}^*(\mathbf{p}), \mathbf{p})}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}), \mathbf{p})}{\partial p_i}.$$

- You can disregard the indirect effects. Graphical intuition.

2 Convexity and concavity

- Function f from $C \subset R^n$ to R is concave if

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$$

for all $x, y \in C$ and for all $t \in [0, 1]$

- Notice: C must be convex set, i.e., if $x \in C$ and $y \in C$, then $tx + (1 - t)y \in C$, for $t \in [0, 1]$

- Function f from $C \subset R^n$ to R is strictly concave if

$$f(tx + (1 - t)y) > tf(x) + (1 - t)f(y)$$

for all $x, y \in C$ and for all $t \in (0, 1)$

- Function f from R^n to R is convex if $-f$ is concave.

- Alternative characterization of convexity.
- A function f , twice differentiable, is concave if and only if for all x the subdeterminants $|H_i|$ of the Hessian matrix have the property $|H_1| \leq 0$, $|H_2| \geq 0$, $|H_3| \leq 0$, and so on.
- For the univariate case, this reduces to $f'' \leq 0$
- For the bivariate case, this reduces to $f''_{x,x} \leq 0$ and $f''_{x,x} * f''_{y,y} - (f''_{x,y})^2 \geq 0$
- A twice-differentiable function is strictly concave if the same property holds with strict inequalities.

- Examples.

1. For which values of $a, b,$ and c is $f(x) = ax^3 + bx^2 + cx + d$ is the function concave over R ?
Strictly concave? Convex?

2. Is $f(x, y) = -x^2 - y^2$ concave?

- For Example 2, compute the Hessian matrix

– $f'_x =$, $f'_y =$

– $f''_{x,x} =$, $f''_{x,y} =$

– $f''_{y,x} =$, $f''_{y,y} =$

– Hessian matrix H :

$$H = \begin{pmatrix} f''_{x,x} = & f''_{x,y} = \\ f''_{y,x} = & f''_{y,y} = \end{pmatrix}$$

- Compute $|H_1| = f''_{x,x}$ and $|H_2| = f''_{x,x} * f''_{y,y} - (f''_{x,y})^2$

- Why are convexity and concavity important?
- Theorem. Consider a twice-differentiable concave (convex) function over $C \subset \mathbb{R}^n$. If the point \mathbf{x}_0 satisfies the first order conditions, it is a global maximum (minimum).
- For the proof, we need to check that the second-order conditions are satisfied.
- These conditions are satisfied by definition of concavity!
- (We have only proved that it is a local maximum)

3 Constrained maximization

- Nicholson, Ch. 2, pp. 39–46
- So far unconstrained maximization on R (or open subsets)
- What if there are constraints to be satisfied?
- Example 1: $\max_{x,y} x * y$ subject to $3x + y = 5$
- Substitute it in: $\max_{x,y} x * (5 - 3x)$
- Solution: $x^* =$
- Example 2: $\max_{x,y} xy$ subject to $x \exp(y) + y \exp(x) = 5$
- Solution: ?

- Graphical intuition on general solution.
- Example 3: $\max_{x,y} f(x, y) = x * y$ s.t. $h(x, y) = x^2 + y^2 - 1 = 0$
- Draw $0 = h(x, y) = x^2 + y^2 - 1$.
- Draw $x * y = K$ with $K > 0$. Vary K
- Where is optimum?
- Where dy/dx along curve $xy = K$ equals dy/dx along curve $x^2 + y^2 - 1 = 0$
- Write down these slopes.

- Idea: Use implicit function theorem.
- Heuristic solution of system

$$\begin{aligned} & \max_{x,y} f(x, y) \\ & \text{s.t. } h(x, y) = 0 \end{aligned}$$

- Assume:
 - continuity and differentiability of h
 - $h'_y \neq 0$ (or $h'_x \neq 0$)
- Implicit function Theorem: Express y as a function of x (or x as function of y)!

- Write system as $\max_x f(x, g(x))$
- f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) * \frac{\partial g(x)}{\partial x} = 0$
- What is $\frac{\partial g(x)}{\partial x}$?
- Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) * (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\begin{array}{l} \max_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n; \mathbf{p}) \\ \text{s.t.} \quad \left\{ \begin{array}{l} h_1(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ h_2(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ \dots \\ h_m(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \end{array} \right. \end{array}$$

with $n > m$. Let $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$ be a local solution to this problem.

- Assume:
 - f and h differentiable at x^*
 - the following Jacobian matrix at \mathbf{x}^* has maximal rank

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \dots & \dots & \dots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Then, there exists a vector $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda})$ maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^m \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case $n = 2, m = 1$.
- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$.

- If \mathbf{x}^* satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1^2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2^2}(\mathbf{x}^*) \end{pmatrix}$$

is positive, then \mathbf{x}^* is a constrained maximum.

- If it is negative, then \mathbf{x}^* is a constrained minimum.
- Why? This is just the Hessian of the Lagrangean L with respect to λ , x_1 , and x_2

- Example 4: $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to x :

- F.o.c. with respect to y :

- F.o.c. with respect to λ :

- Candidates to solution?

- Maxima and minima?