Economics 101A
(Lecture 6, Revised)

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Outline

1. Common Utility Functions

2. Utility Maximization with Lagrangeans

3. Utility maximization – tricky cases

1 Common utility functions

- Nicholson, Ch. 3, pp. 80–84

1. Cobb-Douglas preferences: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

   - $MRS = -\alpha x_1^{a-1} x_2^{1-\alpha} / (1-a) x_1^{\alpha} x_2^{-\alpha} = -\frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$ [REVISED]

2. Perfect substitutes: $u(x_1, x_2) = \alpha x_1 + \beta x_2$

   - $MRS = -\alpha / \beta$

3. Perfect complements: $u(x_1, x_2) = \min(\alpha x_1, \beta x_2)$

   - MRS discontinuous at $x_2 = \frac{\alpha}{\beta} x_1$

4. Constant Elasticity of Substitution: $u(x_1, x_2) = \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$

   - $MRS = -\frac{\alpha}{\beta} \left(\frac{x_1}{x_2}\right)^{\rho - 1}$
   - if $\rho = 1$, then...
   - if $\rho = 0$, then...
   - if $\rho \to -\infty$, then... [CORRECTION]

2 Utility Maximization

- Nicholson, Ch. 4, pp. 91–103
- $X = R^2_+$ (2 goods)
- Consumers: choose bundle $x = (x_1, x_2)$ in X which yields highest utility.
- Constraint: income $= M$
- Price of good 1 $= p_1$, price of good 2 $= p_2$
- Bundle x is feasible if $p_1 x_1 + p_2 x_2 \leq M$
- Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 \leq M$$
$$x_1 \geq 0,\ x_2 \geq 0$$

- Maximization subject to inequality. How do we solve that?

- Trick: u strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

- Budget constraint always satisfied with equality

- Ignore temporarily $x_1 \geq 0$, $x_2 \geq 0$ and check afterwards that they are satisfied for x_1^* and x_2^*.

- Problem becomes

$$\max_{x_1,x_2} u(x_1, x_2)$$
$$s.t.\ p_1x_1 + p_2x_2 - M = 0$$

- $L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 = M)$

- F.o.c.s:

$$\begin{aligned} u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\ p_1x_1 + p_2x_2 - M &= 0 \end{aligned}$$

- Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

- Graphical interpretation.

- Example with CES utility function.

$$\max_{x_1,x_2} \left(\alpha x_1^{\rho} + \beta x_2^{\rho}\right)^{1/\rho}$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- Lagrangean =

- F.o.c.:

- Special case: $\rho = 0$ (Cobb-Douglas)

3 Utility maximization – tricky cases

1. Non-convex preferences. Example:

- Second order conditions:

$$H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix}$$

$$\begin{aligned} |H| &= p_1 \left(-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1}\right) \\ &\quad -p_2 \left(-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1}\right) \\ &= -p_1^2 u''_{x_2,x_2} + 2p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \end{aligned}$$

2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1 * (x_2 + 5)$$
$$s.t.\ p_1 x_1 + p_2 x_2 = M$$

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?

3. Multiplicity of solutions. Example:

- Convex preferences that are not strictly convex

4. Example with CES utility function.

$$\max_{x_1,x_2} \left(\alpha x_1^{\rho} + \beta x_2^{\rho}\right)^{1/\rho}$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- With $\rho > 1$ the interior solution is a minimum!

- Draw indifference curves for $\rho = 1$ (boundary case) and $\rho = 2$

- Can also check using second order conditions