# Economics 101A
# (Lecture 2)

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Outline

1. Univariate Optimization
2. Multivariate Optimization
3. Comparative Statics
4. Implicit function theorem

# 1 Univariate optimization

- **Sufficient condition for a (local) maximum**:

$$\frac{\partial f(x^*)}{\partial x} = 0 \text{ and } \left.\frac{\partial^2 f(x)}{\partial^2 x}\right|_{x^*} < 0$$

- Notice: Sufficient, not necessary condition

- Tricky examples:

  – *Minimum.* $y = x^2$

  – *No maximum.* $y = \exp(x)$ for $x \in (-\infty, +\infty)$

  – *Corner solution.* $y = x$ for $x \in [0, 1]$

# 2 Multivariate optimization

- Nicholson, Ch.2, pp. 26–32

- Function from $R^n$ to $R$: $y = f(x_1, x_2, ..., x_n)$

- Partial derivative with respect to $x_i$:

$$\frac{\partial f(x_1, ..., x_n)}{\partial x_i}$$
$$= \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ...x_n) - f(x_1, ..., x_i, ...x_n)}{h}$$

- Slope along dimension $i$

- Total differential:

$$df = \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 + ... + \frac{\partial f(x)}{\partial x_n} dx_n$$

- One important economic example

- Example 1: Partial derivatives of $y = f(L, K) = L^{.5}K^{.5}$

- $f'_L =$
(marginal productivity of labor)

- $f'_K =$
(marginal productivity of capital)

- $f''_{L,K} = f''_{K,L} =$

Maximization over an open set (like $R$)

- **Necessary condition for maximum** $x^*$ is

$$\frac{\partial f(x^*)}{\partial x_i} = 0 \ \forall i \tag{1}$$

or in vectorial form

$$\nabla f(x) = 0$$

- These are commonly referred to as first order conditions (f.o.c.)

- Sufficient conditions? Define Hessian matrix $H$:

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\ \ldots & \ldots & \ldots & \ldots \\ f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n} \end{pmatrix}$$

- Subdeterminant $|H|_i$ of Matrix $H$ is defined as the determinant of submatrix formed by first $i$ rows and first $i$ columns of matrix $H$.

- Examples.
  - $|H|_1$ is determinant of $f''_{x_1,x_1}$, that is, $f''_{x_1,x_1}$
  - $|H|_2$ is determinant of

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} \\ f''_{x_2,x_1} & f''_{x_2,x_2} \end{pmatrix}$$

- **Sufficient condition for maximum** $x^*$.
  1. $x^*$ must satisy first order conditions;
  2. Subdeterminants of matrix $H$ must have alternating signs, with subdeterminant of $H_1$ negative.

- Case with $n = 2$

- Condition 2 reduces to $f''_{x_1,x_1} < 0$ and $f''_{x_1,x_1} f''_{x_2,x_2} - (f''_{x_1,x_2})^2 > 0$.

- Example 2: $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

- First order condition w/ respect to $x_1$?

- First order condition w/ respect to $x_2$?

- $x_1^*, x_2^* =$

- For which $p_1, p_2$ is it a maximum?

- For which $p_1, p_2$ is it a minimum?

# 3 Comparative statics

- Economics is all about ‘comparative statics’

- What happens to the optimal value if we change one parameter?

- Examples on consumer:

  1. Banana consumption and increase in price of banana?

  2. Banana consumption and increase in price of apple?

- Examples on producer:

1. Banana production and increase in wage of banana growers?

2. Banana production and increase in price of banana?

- Next two sections

# 4 Implicit function theorem

- Consider function $y = g(x, p)$

- Can rewrite as $y - g(x, p) = 0$

- **Implicit function** has form: $h(y, x, p) = 0$

- Often we need to go from implicit to explicit function

- Example 3: $1 - xy - e^{y} = 0$.

- Write $x$ as function of $y$ :

- Write $y$ as function of $x$ :

- **Univariate implicit function theorem (Dini)**: Consider an equation $f(p, x) = 0$, and a point $(p_0, x_0)$ solution of the equation. Assume:

  1. $f$ continuous and differentiable in a neighbourhood of $(p_0, x_0)$;

  2. $f'_x(p_0, x_0) \neq 0$.

- Then:

  1. There is one and only function $x = g(p)$ defined in a neighbourhood of $p_0$ that satisfies $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

  2. The derivative of $g(p)$ is

$$g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}$$

- Example 3 (continued): $1 - xy - e^y = 0$

- Find derivative of $y = g(x)$ implicitely defined for $(x, y) = (1, 0)$

- Assumptions:

  1. Satisfied?

  2. Satisfied?

- Compute derivative

- **Multivariate implicit function theorem (Dini)**: Consider a set of equations ($f_1(p_1, ..., p_n; x_1, ..., x_s) = 0; ...; f_s(p_1, ..., p_n; x_1, ..., x_s) = 0$), and a point ($p_0$,$x_0$) solution of the equation. Assume:

  1. $f_1, ..., f_s$ continuous and differentiable in a neighbourhood of ($p_0$,$x_0$);

     (a) The following Jakobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ evaluated at ($p_0$,$x_0$) has determinant different from 0:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & & \frac{\partial f_1}{\partial x_s} \\ ... & ... & ... \\ \frac{\partial f_s}{\partial x_1} & ... & \frac{\partial f_s}{\partial xs} \end{pmatrix}$$

- Then:

    1. There is one and only set of functions $x$= $\mathbf{g}(p)$ defined in a neighbourhood of $p_0$ that satisfy $\mathbf{f}(p, \mathbf{g}(p)) = \mathbf{0}$ and $\mathbf{g}(p_0)$ =$x_0$;

    2. The partial derivative of $x_i$ with respect to $p_k$ is

$$\frac{\partial g_i}{\partial p_k} = -\frac{\det\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots x_{i-1},p_k,x_{i+1}\ldots,x_s)}\right)}{\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)}$$

- Example 2 (continued): Max $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

- f.o.c. $x_1$ : $2p_1 * x_1 - 2 = 0 = f_1(p,x)$

- f.o.c. $x_2$ : $2p_2 * x_2 - 5 = 0 = f_2(p,x)$

- Comparative statics of $x_1^*$ with respect to $p_1$?

- First compute det $\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

- Then compute $\det\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots x_{i-1},p_k,x_{i+1}\ldots,x_s)}\right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}$$

- Finally, $\frac{\partial x_1}{\partial p_1} =$

- Why did you compute $\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)$ already?

# 5 Envelope Theorem

- You now know how $x_1^*$ varies if $p_1$ varies.

- How does the function $h$ vary at the optimum as $p_1$ varies?

- Differentiate $h(x_1^*(p_1,p_2), x_2^*(p_1,p_2), p_1, p_2)$ with respect to $p_1$ :

$$
\begin{aligned}
& \frac{dh(x_1^*(p_1,p_2), x_2^*(p_1,p_2), p_1, p_2)}{dp_1} \\
= \; & \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_1} * \frac{\partial x_1^*(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \\
& + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_2} * \frac{\partial x_2^*(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \\
& + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial p_1}
\end{aligned}
$$

- Can we say something about the first two terms? They are zero!

- **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(\mathbf{x};\mathbf{p})$ with respect to $x$. Consider then the function $f$ at the optimum, that is, $f(\mathbf{x}^*(\mathbf{p}),\mathbf{p})$. The total differential of this function with respect to $p_i$ equals the partial derivative with respect to $p_i$:

$$\frac{df(\mathbf{x}^*(\mathbf{p}),\mathbf{p})}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}),\mathbf{p})}{\partial p_i}.$$

- You can disregard the indirect effects. Graphical intuition.

# 6 Next Class

- Next class:
  - Convexity and Concavity
  - Constrained Maximization
  - Envelope Theorem II

- Going toward:
  - Preferences
  - Utility Maximization (where we get to apply maximization techniques the first time)