Econ 101A – Problem Set 2
Due in class on Tu September 28. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated in lectures 4 to 7. It is focused on preferences, utility functions, and utility maximization. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Addictive goods. (23 points) In this exercise, we propose a generalization of Cobb-Douglas preferences that incorporates the concept of reference point. We use it to model the consumption of addictive goods. Consider the following utility function:

$$u(x_1, x_2; r_1) = (x_1 - r_1)^{\alpha} {x_2}^{\beta}$$

with $\alpha + \beta = 1$, $0 < \alpha < 1$, $0 < \beta < 1$, and $r_1 > 0$. Notice that the above utility is only defined for $x_1 \geq r_1$ and $x_2 \geq 0$. Assume that for $x_1 < r_1$ or $x_2 < 0$ the utility is zero. Good x_1 is an addictive good with addiction level r_1. Examples of addictive goods are alcohol, drugs or... chocolate. The more you have consumed of these goods in the past, the higher the addiction level r_1.

1. Draw an approximate map of indifference curves for the case $\alpha = \beta = .5$. (2 points)
2. How does the utility function change as r_1 changes? In other words, compute $\partial u(x_1, x_2; r_1)/\partial r_1$. Why is this term negative? [Hint: If I have gotten used to drinking a lot of alcohol, my utility of drinking three bottles of beer...] (3 points)
3. Compute now the marginal utility with respect to x_1. In other words, compute $\partial u(x_1, x_2; r_1)/\partial x_1$ for $x_1 > r_1$. How does this marginal utility change as r_1 changes? In other words, compute $\partial^2 u(x_1, x_2; r_1)/\partial x_1 \partial r_1$ for $x_1 > r_1$. Why is this term positive? [Hint: If I have gotten used to drinking a lot of alcohol, my desire to drink one *more* bottle of beer...] (3 points)
4. Consider now the maximization subject to a budget constraint. The agent maximizes
$$\begin{aligned} \max_{x_1, x_2} u(x_1, x_2) &= (x_1 - r_1)^{\alpha} {x_2}^{\beta} \\ \text{s.t. } p_1 x_1 + p_2 x_2 &= M. \end{aligned}$$
Write down the Lagrangean function. (1 point)
5. Write down the first order conditions for this problem with respect to x_1, x_2, and λ. (1 point)
6. Solve explicitely for x_1^* and x_2^* as a function of p_1, p_2, M, r_1, α, and β. [You do not have to check the second order conditions. I guarantee that they are satisfied :-), provided that the condition in point 7 is satified] (3 points)
7. What is the minimum level of income in order for the solution to make sense, i.e., so that $x_1^* \geq r_1$ and $x_2^* \geq 0$? (for a lower level of income the agent would have zero utility) (2 points)
8. Is good x_1 a normal good, i.e., is $\partial x_1^*/\partial M > 0$ for all values of M above the minimum level of income in point 7? (2 points)
9. Compute the change in x_1^* as r_1 varies. In order to do so, use directly the expressions that you obtained in point 6, and differentiate x_1^* with respect to r_1. Does your result make sense? Why do I consumer more of good 1 if I am more addicted to it (higher r_1)? (2 points)
10. Compute the change in x_2^* as r_1 varies: differentiate x_2^* with respect to r_1. Does your result make sense? Think of the case of drug addicts that spend virutally all of their income into buying drugs. (2 points)

11. Use the envelope theorem to calculate $\partial u(x_1^*(p_1, p_2, M; r_1), x_2^*(p_1, p_2, M; r_1); r_1)/\partial r_1$. What happens to utility at the optimum as the level of addition increases? (2 points)

Problem 2. Quasi-linear preferences (25 points) In economics, it is often convenient to write the utility function in a quasi-linear form. These utility functions have the following form:

$$u(x_1, x_2) = \phi(x_1) + x_2$$

with $\phi'(x) > 0$, and $\phi''(x) < 0$. These preferences are called quasi-linear because the utility function is linear in good 2. In this exercise we explore several convenient properties of this utility function. We will do so at first without assuming a particular functional form for $\phi(x)$.

Consider the maximization subject to a budget constraint. The agent maximizes

$$\max_{x_1, x_2} \ \phi(x_1) + x_2$$
$$\text{s.t. } p_1 x_1 + p_2 x_2 = M$$

with $p_1 > 0$, $p_2 > 0, M > 0$.

1. Write down the Lagrangean function (1 point)
2. Write down the first order conditions for this problem with respect to x_1, x_2, and λ. (1 point)
3. What do the first order conditions tell you regarding the value of λ? **(Hint: Use the first order condition with respect to x_2)** Does the value of λ depend on p_1 or M? (usually it does) Why is this the case? Think of λ as the marginal utility of wealth. (3 points)
4. Plug the value of λ into the first order condition for x_1. You now have an equation that implicitely defines x_1^* as a function of the parameters p_1, p_2, M. Does the optimal quantity of x_1^* depend on income M? Is good 1 a normal good ($\partial x_1^*/\partial M > 0$), an inferior good ($\partial x_1^*/\partial M < 0$), or a neutral good ($\partial x_1^*/\partial M = 0$)? [If $\partial x_1^*/\partial M = 0$, we say that it is a neutral good, i.e., that there is no income effect] (5 points)
5. Use the implicit function theorem to compute $\partial x_1^*/\partial p_1$ from the first order condition with respect to x_1 (remember, you have already substituted for the value of λ). You should find that $\partial x_1^*/\partial p_1 < 0$. You should know this already from the answer to point 4. Why? (Hint: think about the Slutzky equation) (5 points)
6. Continue now under the assumption $u(x_1, x_2) = x_1^{1/2} + x_2$. Explicitely solve for x_1^* and then, using the budget constraint, solve for x_2^*. (2 points)
7. Under what conditions for p_1, p_2, and M is $x_2^* \geq 0$? (2 points)
8. The indifference curves satisfy equation $x_1^{1/2} + x_2 = \overline{u}$ or $x_2 = \overline{u} - x_1^{1/2}$. Draw a map of indifference curves in the space (x_1, x_2). What is special about this indifference curves? (compare them, for example, to the ones for Cobb-Douglas preferences) (3 points)
9. Write down two budget lines: for $(p_1 = 1, p_2 = 1, M = 1)$ and for $(p_1 = 1, p_2 = 1, M = 2)$. Find graphically the optimal consumption bundles by tangency of the budget set and the indifference curve. You should find that $x_1^*(1, 1, 1) = x_1^*(1, 1, 2)$. This means that there is no income effect in good 1. The increase in income goes all toward good 2. This should be a graphical confirmation of what you found at point 4 (3 points)

Problem 3. Expenditure minimization–Tricky cases. (11 points) Here are two expenditure minimization problems in which you are not supposed to take derivatives. Use your intuition and graphical methods. The solution is similar to the ones that we explored in class for the case of utility maximization.

$$\begin{aligned} &\min_{x_1,x_2} p_1x_1 + p_2x_2 \\ &\text{s.t. } u(x_1,x_2) = \overline{u}, \end{aligned}$$

1. Assume $u(x_1, x_2) = \min(x_1, x_2)$.
   (a) What is the solution for h_1^* and h_2^*, that is, for the Hicksian compensated demand functions? (do not write the Lagrangean, try graphically). (4 points)
   (b) Show that the Hicksian compensated demand function does not depend on p_1 or p_2, that is, $\partial h_i^*(\mathbf{p}, \overline{u})/\partial p_i = \partial h_i^*(\mathbf{p}, \overline{u})/\partial p_j = 0$. In this case, the substitution effect of a change in price is null. (3 points)
2. Assume $u(x_1, x_2) = x_1^2 + x_2^2$. What is the solution for h_1^* and h_2^*, that is, for the Hicksian compensated demand functions? (do not write the Lagrangean, try graphically). (4 points)